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EXAM TWO, MTH 205, SPRING 008
THIS IS THE REAL TEST

AYMAN BADAWI

ID NUMBER 2712 Name: Hasnaa Rabbat. Score =

QUESTION 1. 20 points *Solve* $y^{(2)} - 6y' + 9y = \frac{e^{3x}}{x^2} + e^{-3x}$

$y'' - 6y' + 9y = 0$ (homogeneous equation).

$y = e^{mx}$

$m^2 - 6m + 9 = (m-3)^2 \Rightarrow m = 3$

$y_c = c_1 e^{3x} + c_2 x e^{3x}$.

(undetermined coefficient)

$y_{p1} = A_0 e^{-3x}$

$A 9e^{-3x} + A18e^{-3x} + A9e^{-3x} = e^{-3x}$

$A36e^{-3x} = e^{-3x} \Rightarrow A = 1/36$ $y_{p1} = \frac{e^{-3x}}{36}$

Good OK

$y_{p2} = f_1 e^{3x} + f_2 x e^{3x}$ (variant method).

$f_1' e^{3x} + f_2' x e^{3x} = 0$

$f_1'(3e^{3x}) + f_2' e^{3x} + 3xe^{3x} f_2' = \frac{e^{3x}}{x^2} \Rightarrow f_1'(3) + (1+3x) f_2' = \frac{1}{x^2}$

$b = \det \begin{bmatrix} e^{3x} & xe^{3x} \\ 3 & 1+3x \end{bmatrix} = e^{3x} + 3xe^{3x} - 3xe^{3x} = e^{3x}$

$f_1' = \frac{\det \begin{bmatrix} 0 & xe^{3x} \\ \frac{1}{x^2} & 1+3x \end{bmatrix}}{b} = \frac{-1e^{3x}}{x} \cdot \frac{1}{e^{3x}} = \frac{-1}{x}$ $f_1 = \int f_1' = -\ln x$

$f_2' = \frac{\det \begin{bmatrix} e^{3x} & 0 \\ 3 & \frac{1}{x^2} \end{bmatrix}}{b} = \frac{e^{3x}}{x^2} \cdot \frac{1}{e^{3x}} = \frac{1}{x^2}$ $f_2 = \frac{-1}{x} = \int f_2'$

$\Rightarrow y_{p2} = -\ln x\, e^{3x} + \left(\frac{-1}{x}\right) x e^{3x}$

Good

QUESTION 2. 20 points *Solve* $y^{(2)} + \frac{1}{x}y' = \cancel{\frac{1}{x^3}}\ \frac{1}{x^2}$

$y'' + \frac{1}{x}y' = 0$ (cauchy euler).

$y = x^m \quad y' = mx^{m-1} \quad y'' = m(m-1)x^{m-2}$

$m(m-1)x^{m-2} + \frac{1}{x}mx^{m-1} = 0$

$\Rightarrow m^2 - m + m = 0 = m^2.$

$y_c = c_1x^0 + c_2x^0\ln x = c_1 + c_2\ln x$

$y_p = f_1(1) + f_2(\ln x)$

$f_1' + f_2'\ln x = 0$

$0 + \frac{f_2'}{x} = \frac{1}{x^2} \Rightarrow f_2' = \frac{1}{x} \Rightarrow f_2 = \ln x$

$f_1' + \frac{\ln x}{x} = 0 \Rightarrow f_1' = -\frac{\ln x}{x} \qquad f_1 = -\int \frac{\ln x}{x} = -\int u\,du$

$= -\frac{(\ln x)^2}{2}.$

$\underline{y_g} = c_1 + c_2\ln x - \frac{(\ln x)^2}{2} + (\ln x)^2$

$= c_1 + c_2\ln x + \frac{(\ln x)^2}{2}.$

QUESTION 3. 20 points *Solve* $y' + cos(2x)y = \frac{cos(2x)}{y}$ $= \cos(2x)\, y^{-1}$

Bernoulli:

$u = y^{1-(-1)} = y^2$

$y = u^{1/2}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} u^{-1/2} u'$

$\frac{1}{2} u^{-1/2} u' + \cos(2x)\, u^{1/2} = \cos 2x \left(\frac{1}{u^{1/2}}\right)$

$\frac{1}{2} u' + \cos(2x)\, u = \cos 2x$

$u' + (2\cos 2x)\, u = 2\cos 2x$

$F(x) = 2\cos 2x$

$p(x) = 2\cos 2x$

$u = \frac{\int I \cdot F(x)}{I}$

$I = e^{\int 2\cos 2x\, dx}$

$= e^{\sin 2x}$

$= \frac{\int \cancel{2} \cos 2x\, e^{\sin 2x}}{e^{\sin 2x}}$

$\frac{\int e^{s}\, du}{e^{\sin 2x}} = \frac{e^{\sin 2x} + c}{e^{\sin 2x}} = u = 1 + \frac{c}{e^{\sin 2x}}$

$= 1 + Ce^{-\sin 2x}$

$y = \sqrt{u} = \sqrt{\frac{e^{\sin 2x}}{c + e^{\sin 2x}}}$?

what !!

-1

QUESTION 4. 20 points *Solve* $y^{(2)} - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y = \frac{2}{x^2-1}$ *if* $y = x$ *is a solution to the associated homogeneous system.*

Reduction to first order: (homogeneous).

$e^{-\int \frac{-2x}{x^2+1}dx} = e^{\ln|x^2+1|} = |x^2+1| = (x^2+1)$ always +ve

$y_2 = y_1 \int \frac{x^2+1}{y_1^2}dx = x\int \frac{x^2+1}{x^2}dx = x\int 1+\frac{1}{x^2}dx$

$= x\left(x - \frac{1}{x}\right) = \underline{x^2-1}$

$y_c = c_1(x) + c_2(x^2-1)$

$y_p = f_1(x) + f_2(x^2-1)$

$f_1'(x) + f_2'(x^2-1) = 0$

$f_1' + f_2'(2x) = \frac{2}{x^2-1}$

$b = \det\begin{bmatrix} x & x^2-1 \\ 1 & 2x \end{bmatrix} = 2x^2 - x^2+1 = \underline{x^2+1}$

$f_1' = \frac{\det\begin{bmatrix} 0 & x^2-1 \\ \frac{2}{x^2-1} & 2x \end{bmatrix}}{b} = \frac{0-2(x^2-1)}{x^2-1}\cdot\frac{1}{x^2+1}$ $f_1 = \underline{-2\tan^{-1}(x)}$

$f_2' = \frac{\det\begin{bmatrix} x & 0 \\ 1 & \frac{2}{x^2-1} \end{bmatrix}}{b} = \frac{\frac{2x}{x^2-1}}{x^2+1} = \frac{2x}{(x^2-1)(x^2+1)}$

Good

$u = x^2$

$f_2 = \int f_2' = \int \frac{1}{(u-1)(u+1)}du$

$f_2 = \int \frac{1}{2(u-1)} - \frac{1}{2(u+1)}du = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|$

$= \underline{\frac{1}{2}\ln|x^2-1| - \frac{1}{2}\ln|x^2+1|}$

+3

$y_p = -2x\tan^{-1}x + (x^2-1)\left[\frac{1}{2}\ln|x^2-1| - \frac{1}{2}\ln|x^2+1|\right]$

QUESTION 5. 20 points *A tank initially contains 20 gallons of Fresh WATER (i.e. when t = 0, amount of salt is zero). A mixture containing 0.5 bound of salt per gallon is poured into the tank at rate of 2 gallons per minute, while the mixture leaves the tank at rate 4 gallons per minute.*

a) Find the amount of salt in the tank at any time t.

b) When will the tank be empty?

c) Find the concentration of the salt in the tank at t = 9.5 minutes.

DEPARTMENT OF MATHEMATICS & STATISTICS, AMERICAN UNIVERSITY OF SHARJAH, P.O. BOX 26666, SHARJAH, UNITED ARAB EMIRATES

E-mail address: abadawi@aus.edu, www.ayman-badawi.com

$V_0 = 20$ gallons $= V(0)$ $\quad A(0)=0$

$C_{in} = 0.5$ pound/gallon $\quad R_{in} = 2$ gallons/min

$R_{out} = 4$ gallons/min.

a) $\frac{dA}{dt} = R_{in} \cdot C_{in} - R_{out} C_{out}$ $\quad V(t) = 20 + (R_{in} - R_{out})t = 20 - 2t.$

$A' = 0.5(2) - \frac{A(t)}{20-2t} \cdot 4$ $\quad C_{out} = \frac{A(t)}{V} = \frac{A(t)}{20-2t}$

$\Rightarrow A' + A(t)\left(\frac{4}{20-2t}\right) = 1$

1st order linear:

$I = e^{\int \frac{4}{20-2t} dt} = e^{\int \frac{2}{10-t} dt} = e^{-2\ln|10-t|}$ (if $t > 10$) $\quad I = (10-t)^{-2}$

$A = \frac{\int (10-t)^{-2} dt}{(10-t)^{-2}} = \frac{+1(10-t)^{-1} + c}{(10-t)^{-2}} = (10-t) + c(10-t)^2 = A(t).$

$A(0) = 0 \Rightarrow 10 + c(10)^2 = 0 \Rightarrow c = -\frac{1}{10} = -0.1$

$A(t) = 10 - t - 0.1(10-t)^2$

b) when $V(t) = 0$ $\quad 0 = 20 - 2t \Rightarrow 20 = 2t \Rightarrow t = 10$ minutes.

At 10 minutes, the tank will be empty.

c) $A(9.5) = 10 - 9.5 - 0.1(10-9.5)^2 = 0.5 - 0.025 = 0.475$ pounds