Communications in Algebra®, 42: 1–14, 2014

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927872.2012.707262

ON THE ANNIHILATOR GRAPH OF A COMMUTATIVE RING

Ayman Badawi
Department of Mathematics and Statistics, American University of Sharjah, Sharjah, UAE

Let R be a commutative ring with nonzero identity, $Z(R)$ be its set of zero-divisors, and if $a \in Z(R)$, then let $ann_R(a) = \{d \in R \mid da = 0\}$. The annihilator graph of R is the (undirected) graph $AG(R)$ with vertices $Z(R)^* = Z(R)\setminus\{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. It follows that each edge (path) of the zero-divisor graph $\Gamma(R)$ is an edge (path) of $AG(R)$. In this article, we study the graph $AG(R)$. For a commutative ring R, we show that $AG(R)$ is connected with diameter at most two and with girth at most four provided that $AG(R)$ has a cycle. Among other things, for a reduced commutative ring R, we show that the annihilator graph $AG(R)$ is identical to the zero-divisor graph $\Gamma(R)$ if and only if R has exactly two minimal prime ideals.

Key Words: Annihilator graph; Annihilator ideal; Zero-divisor graph.

2000 Mathematics Subject Classification: Primary 13A15; Secondary 05C99. Q1

1. INTRODUCTION

Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. Recently, there has been considerable attention in the literature to associating graphs with algebraic structures (see [8, 11–14]). Probably the most attention has been to the *zero-divisor graph* $\Gamma(R)$ for a commutative ring R. The set of vertices of $\Gamma(R)$ is $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The concept of a zero-divisor graph goes back to Beck [6], who let all elements of R be vertices and was mainly interested in colorings. The zero-divisor graph was introduced by David F. Anderson and Paul S. Livingston in [3], where it was shown, among other things, that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $\mathrm{gr}(\Gamma(R)) \in \{3, 4, \infty\}$. For a recent survey article on zero-divisor graphs, see [5]. In this article, we introduce the *annihilator graph* $AG(R)$ for a commutative ring R. Let $a \in Z(R)$ and let $ann_R(a) = \{r \in R \mid ra = 0\}$. The annihilator graph of R is the (undirected) graph $AG(R)$ with vertices $Z(R)^* = Z(R)\setminus\{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. It follows that each edge (path) of $\Gamma(R)$ is an edge (path) of $AG(R)$.

Received December 27, 2011; Revised June 14, 2012. Communicated by S. Bazzoni.
Address correspondence to Ayman Badawi, Department of Mathematics and Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, UAE; E-mail: abadawi@aus.edu

In the second section, we show that $AG(R)$ is connected with diameter at most two (Theorem 2.2). If $AG(R)$ is not identical to $\Gamma(R)$, then we show that $gr(AG(R))$ (i.e., the length of a smallest cycle) is at most four (Corollary 2.11). In the third section, we determine when $AG(R)$ is identical to $\Gamma(R)$. For a reduced commutative ring R, we show that $AG(R)$ is identical to $\Gamma(R)$ if and only if R has exactly two distinct minimal prime ideals (Theorem 3.6). Among other things, we determine when $AG(R)$ is a complete graph, a complete bipartite graph, or a star graph.

Let Γ be a (undirected) graph. We say that Γ is *connected* if there is a path between any two distinct vertices. For vertices x and y of Γ, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path). Then the *diameter* of Γ is $diam(\Gamma) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } \Gamma\}$. The *girth* of Γ, denoted by $gr(\Gamma)$, is the length of a shortest cycle in Γ ($gr(\Gamma) = \infty$ if Γ contains no cycles).

A graph Γ is *complete* if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K^n (we allow n to be an infinite cardinal). A *complete bipartite graph* is a graph Γ which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex sets is a singleton, then we call Γ a *star graph*. We denote the complete bipartite graph by $K^{m,n}$, where $|A| = m$ and $|B| = n$ (again, we allow m and n to be infinite cardinals); so a star graph is a $K^{1,n}$ and $K^{1,\infty}$ denotes a star graph with infinitely many vertices. Finally, let $\overline{K}^{m,3}$ be the graph formed by joining $\Gamma_1 = K^{m,3}$ ($= A \cup B$ with $|A| = m$ and $|B| = 3$) to the star graph $\Gamma_2 = K^{1,m}$ by identifying the center of Γ_2 and a point of B.

Throughout, R will be a commutative ring with nonzero identity, $Z(R)$ its set of zero-divisors, $Nil(R)$ its set of nilpotent elements, $U(R)$ its group of units, $T(R)$ its total quotient ring, and $Min(R)$ its set of minimal prime ideals. For any $A \subseteq R$, let $A^* = A \setminus \{0\}$. We say that R is *reduced* if $Nil(R) = \{0\}$ and that R is *quasi-local* if R has a unique maximal ideal. The distance between two distinct vertices a, b of $\Gamma(R)$ is denoted by $d_{\Gamma(R)}(a, b)$. If $AG(R)$ is identical to $\Gamma(R)$, then we write $AG(R) = \Gamma(R)$; otherwise, we write $AG(R) \neq \Gamma(R)$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo n, respectively. Any undefined notation or terminology is standard, as in [9] or [7].

2. BASIC PROPERTIES OF *AG*(*R*)

In this section, we show that $AG(R)$ is connected with diameter at most two. If $AG(R) \neq \Gamma(R)$, we show that $gr(AG(R)) \in \{3, 4\}$. If $|Z(R)^*| = 1$ for a commutative ring R, then R is ring-isomorphic to either Z_4 or $Z_2[X]/(X^2)$ and hence $AG(R) = \Gamma(R)$. Since commutative rings with exactly one nonzero zero-divisor are studied in [2, 3, 10], throughout this article we only consider commutative rings with at least two nonzero zero-divisors.

We begin with a lemma containing several useful properties of $AG(R)$.

Lemma 2.1. *Let R be a commutative ring.*

(1) *Let x, y be distinct elements of $Z(R)^*$. Then $x - y$ is not an edge of $AG(R)$ if and only if $ann_R(xy) = ann_R(x)$ or $ann_R(xy) = ann_R(y)$.*

(2) *If $x - y$ is an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, then $x - y$ is an edge of $AG(R)$. In particular, if P is a path in $\Gamma(R)$, then P is a path in $AG(R)$.*
(3) *If $x - y$ is not an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$, then $ann_R(x) \subseteq ann_R(y)$ or $ann_R(y) \subseteq ann_R(x)$.*
(4) *If $ann_R(x) \nsubseteq ann_R(y)$ and $ann_R(y) \nsubseteq ann_R(x)$ for some distinct $x, y \in Z(R)^*$, then $x - y$ is an edge of $AG(R)$.*
(5) *If $d_{\Gamma(R)}(x, y) = 3$ for some distinct $x, y \in Z(R)^*$, then $x - y$ is an edge of $AG(R)$.*
(6) *If $x - y$ is not an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$, then there is a $w \in Z(R)^* \backslash \{x, y\}$ such that $x - w - y$ is a path in $\Gamma(R)$, and hence $x - w - y$ is also a path in $AG(R)$.*

Proof. (1) Suppose that $x - y$ is not an edge of $AG(R)$. Then $ann_R(xy) = ann_R(x) \cup ann_R(y)$ by definition. Since $ann_R(xy)$ is a union of two ideals, we have $ann_R(xy) = ann_R(x)$ or $ann_R(xy) = ann_R(y)$. Conversely, suppose that $ann_R(xy) = ann_R(x)$ or $ann_R(xy) = ann_R(y)$. Then $ann_R(xy) = ann_R(x) \cup ann_R(y)$, and thus $x - y$ is not an edge of $AG(R)$.

(2) Suppose that $x - y$ is an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Then $xy = 0$ and hence $ann_R(xy) = R$. Since $x \neq 0$ and $y \neq 0$, $ann_R(x) \neq R$ and $ann_R(y) \neq R$. Thus $x - y$ is an edge of $AG(R)$. The "in particular" statement is now clear.

(3) Suppose that $x - y$ is not an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$. Then $ann_R(x) \cup ann_R(y) = ann_R(xy)$. Since $ann_R(xy)$ is a union of two ideals, we have $ann_R(x) \subseteq ann_R(y)$ or $ann_R(y) \subseteq ann_R(x)$.

(4) This statement is now clear by (3).

(5) Suppose that $d_{\Gamma(R)}(x, y) = 3$ for some distinct $x, y \in Z(R)^*$. Then $ann_R(x) \nsubseteq ann_R(y)$ and $ann_R(y) \nsubseteq ann_R(x)$. Hence $x - y$ is an edge of $AG(R)$ by (4).

(6) Suppose that $x - y$ is not an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$. Then there is a $w \in ann_R(x) \cap ann_R(y)$ such that $w \neq 0$ by (3). Since $xy \neq 0$, we have $w \in Z(R)^* \backslash \{x, y\}$. Hence $x - w - y$ is a path in $\Gamma(R)$, and thus $x - w - y$ is a path in $AG(R)$ by (2). □

In view of (6) in the preceding lemma, we have the following result.

Theorem 2.2. *Let R be a commutative ring with $|Z(R)^*| \geq 2$. Then $AG(R)$ is connected and $diam(AG(R)) \leq 2$.*

Lemma 2.3. *Let R be a commutative ring, and let x, y be distinct nonzero elements. Suppose that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. If there is a $w \in ann_R(xy) \backslash \{x, y\}$ such that $wx \neq 0$ and $wy \neq 0$, then $x - w - y$ is a path in $AG(R)$ that is not a path in $\Gamma(R)$, and hence $C : x - w - y - x$ is a cycle in $AG(R)$ of length three and each edge of C is not an edge of $\Gamma(R)$.*

Proof. Suppose that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$. Then $xy \neq 0$. Assume there is a $w \in ann_R(xy) \backslash \{x, y\}$ such that $wx \neq 0$ and $wy \neq 0$. Since $y \in ann_R(xw) \backslash (ann_R(x) \cup ann_R(w))$, we conclude that $x - w$ is an edge of $AG(R)$. Since $x \in ann_R(yw) \backslash (ann_R(y) \cup ann_R(w))$, we conclude that $y - w$ is an edge of

$AG(R)$. Hence $x - w - y$ is a path in $AG(R)$. Since $xw \neq 0$ and $yw \neq 0$, we have $x - w - y$ is not a path in $\Gamma(R)$. It is clear that $x - w - y - x$ is a cycle in $AG(R)$ of length three and each edge of C is not an edge of $\Gamma(R)$. □

Theorem 2.4. *Let R be a commutative ring. Suppose that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. If $xy^2 \neq 0$ and $x^2y \neq 0$, then there is a $w \in Z(R)^*$ such that $x - w - y$ is a path in $AG(R)$ that is not a path in $\Gamma(R)$, and hence $C : x - w - y - x$ is a cycle in $AG(R)$ of length three and each edge of C is not an edge of $\Gamma(R)$.*

Proof. Suppose that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$. Then $xy \neq 0$ and there is a $w \in ann_R(xy) \setminus (ann_R(x) \cup ann_R(y))$. We show $w \notin \{x, y\}$. Assume $w \in \{x, y\}$. Then either $x^2y = 0$ or $y^2x = 0$, which is a contradiction. Thus $w \notin \{x, y\}$. Hence $x - w - y$ is the desired path in $AG(R)$ by Lemma 2.3. □

Corollary 2.5. *Let R be a reduced commutative ring. Suppose that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Then there is a $w \in ann_R(xy) \setminus \{x, y\}$ such that $x - w - y$ is a path in $AG(R)$ that is not a path in $\Gamma(R)$, and hence $C : x - w - y - x$ is a cycle in $AG(R)$ of length three and each edge of C is not an edge of $\Gamma(R)$.*

Proof. Suppose that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Since R is reduced, we have $(xy)^2 \neq 0$. Hence $xy^2 \neq 0$ and $x^2y \neq 0$, and thus the claim is now clear by Theorem 2.4. □

In light of Corollary 2.5, we have the following result.

Theorem 2.6. *Let R be a reduced commutative ring, and suppose that $AG(R) \neq \Gamma(R)$. Then $gr(AG(R)) = 3$. Furthermore, there is a cycle C of length three in $AG(R)$ such that each edge of C is not an edge of $\Gamma(R)$.*

In view of Theorem 2.4, the following is an example of a nonreduced commutative ring R where $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, but every path in $AG(R)$ of length two from x to y is also a path in $\Gamma(R)$.

Example 2.7. Let $R = \mathbb{Z}_8$. Then $2 - 6$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$. Now $2 - 4 - 6$ is the only path in $AG(R)$ of length two from 2 to 6 and it is also a path in $\Gamma(R)$. Note that $AG(R) = K^3$, $\Gamma(R) = K^{1,2}$, $gr(\Gamma(R)) = \infty$, $gr(AG(R)) = 3$, $diam(\Gamma(R)) = 2$, and $diam(AG(R)) = 1$.

The following is an example of a nonreduced commutative ring R such that $AG(R) \neq \Gamma(R)$ and if $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, then there is no path in $AG(R)$ of length two from x to y.

Example 2.8.

(1) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and let $a = (0, 1)$, $b = (1, 2)$, and $c = (0, 3)$. Then $a - b$ and $c - b$ are the only two edges of $AG(R)$ that are not edges of $\Gamma(R)$, but there is

no path in $AG(R)$ of length two from a to b and there is no path in $AG(R)$ of length two from c to b. Note that $AG(R) = K^{2,3}$, $\Gamma(R) = \overline{K}^{1,3}$, $gr(AG(R)) = 4$, $gr(\Gamma(R)) = \infty$, $diam(AG(R) = 2$, and $diam(\Gamma(R)) = 3$.

(2) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$. Let $x = X + (X^2) \in \mathbb{Z}_2[X]/(X^2)$, $a = (0, 1)$, $b = (1, x)$, and $c = (0, 1 + x)$. Then $a - b$ and $c - b$ are the only two edges of $AG(R)$ that are not edges of $\Gamma(R)$, but there is no path in $AG(R)$ of length two from a to b and there is no path in $AG(R)$ of length two from c to b. Again, note that $AG(R) = K^{2,3}$, $\Gamma(R) = \overline{K}^{1,3}$, $gr(AG(R)) = 4$, $gr(\Gamma(R)) = \infty$, $diam(AG(R) = 2$, and $diam(\Gamma(R)) = 3$.

Theorem 2.9. *Let R be a commutative ring and suppose that $AG(R) \neq \Gamma(R)$. Then the following statements are equivalent*:

(1) $gr(AG(R)) = 4$;
(2) $gr(AG(R)) \neq 3$;
(3) *If $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, then there is no path in $AG(R)$ of length two from x to y*;
(4) *There are some distinct $x, y \in Z(R)^*$ such that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ and there is no path in $AG(R)$ of length two from x to y*;
(5) *R is ring-isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.*

Proof. (1) $\Rightarrow$ (2). No comments.

(2) $\Rightarrow$ (3). Suppose that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Since $gr(AG(R)) \neq 3$, there is no path in $AG(R)$ of length two from x to y.

(3) $\Rightarrow$ (4). Since $AG(R)) \neq \Gamma(R)$ by hypothesis, there are some distinct $x, y \in Z(R)^*$ such that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$, and hence there is no path in $AG(R)$ of length two from x to y by (3).

(4) $\Rightarrow$ (5). Suppose there are some distinct $x, y \in Z(R)^*$ such that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ and there is no path in $AG(R)$ of length two from x to y. Then $ann_R(x) \cap ann_R(y) = \{0\}$. Since $xy \neq 0$ and $ann_R(x) \cap ann_R(y) = \{0\}$, by Lemma 2.3 we conclude that $ann_R(xy) = ann_R(x) \cup ann_R(y) \cup \{y\}$ such that $y^2 \neq 0$ or $ann_R(xy) = ann_R(x) \cup ann_R(y) \cup \{x\}$ such that $x^2 \neq 0$ (note that if $\{x, y\} \subseteq ann_R(xy)$, then $x - xy - y$ is a path in $AG(R)$ of length two). Without lost of generality, we may assume that $ann_R(xy) = ann_R(x) \cup ann_R(y) \cup \{y\}$ and $y^2 \neq 0$. Let a be a nonzero element of $ann_R(x)$ and b be a nonzero element of $ann_R(y)$. Since $ann_R(x) \cap ann_R(y) = \{0\}$, we have $a + b \in ann_R(xy) \backslash (ann_R(x) \cup ann_R(y))$, and hence $a + b = y$. Thus $|ann_R(x)| = |ann_R(y)| = 2$. Since $xy^2 = 0$, we have $ann_R(x) = \{0, y^2\}$ and $ann_R(y) = \{0, xy\}$. Since $y^2 + xy = y$, we have $(y^2 + xy)^2 = y^2$. Since $xy^3 = 0$ and $xy^2 = x^2y^2 = 0$, we have $(y^2 + xy)^2 = y^2$ implies that $y^4 = y^2$. Since $y^2 \neq 0$ and $y^4 = y^2$, we have y^2 is a nonzero idempotent of R. Hence $ann_R(xy) = ann_R(x) \cup ann_R(y) \cup \{y\} = \{0, y^2, xy, y\}$. Thus $ann_R(xy) \subseteq yR$ and since $yR \subseteq ann_R(xy)$, we conclude $ann_R(xy) = yR = \{0, y^2, xy, y\}$. Since $y^2 + xy = y$ and $y^4 = y^2$, we have $(y^2 + xy)^3 = y^3$ and hence $y^3 = y^2$. Thus $y^2R = y(yR) = \{0, y^2\}$. Since y^2 is a nonzero idempotent of R and y^2R is a ring with two elements, we conclude that y^2R is ring-isomorphic to $\mathbb{Z}_2$. Let $f \in ann_R(y^2)$. Then $y^2f = y(yf) =$

0, and thus $yf \in ann_R(y)$. Hence either $yf = 0$ or $yf = yx$. Suppose $yf = 0$. Since $ann_R(y) = \{0, xy\}$, either $f = 0$ or $f = xy$. Suppose $yf = yx$. Then $y(f - x) = 0$, and thus $f - x = 0$ or $f - x = xy$. Hence $f = x$ or $f = x + xy$. It is clear that $0, x, xy, x + xy$ are distinct elements of R and thus $ann_R(y^2) = \{0, x, xy, x + xy\}$. Since $ann_R(y^2) = (1 - y^2)R$, we have $(1 - y^2)R = \{0, x, xy, x + xy\}$. Since $(1 - y^2)R$ is a ring with four elements, we conclude that $(1 - y^2)R$ is ring-isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ or F_4 or $\mathbb{Z}_2[X]/(X^2)$. Since $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ and there is no path in $AG(R)$ of length two from x to y by hypothesis, we conclude that R is non-reduced by Corollary 2.5. Since R is ring-isomorphic to $y^2R \times (1 - y^2)R$ and non-reduced, we conclude that R is ring-isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.

(5) ⇒ (1). See Example 2.8. □

Corollary 2.10. *Let R be a commutative ring such that $AG(R) \neq \Gamma(R)$, and assume that R is not ring-isomorphic to $\mathbb{Z}_2 \times B$, where $B = \mathbb{Z}_4$ or $B = \mathbb{Z}_2[X]/(X^2)$. If E is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$, then E is an edge of a cycle of length three in $AG(R)$.*

Corollary 2.11. *Let R be a commutative ring such that $AG(R) \neq \Gamma(R)$. Then $gr(AG(R)) \in \{3, 4\}$.*

Proof. This result is a direct implication of Theorem 2.9. □

3. WHEN IS $AG(R)$ IDENTICAL TO $\Gamma(R)$?

Let R be a commutative ring such that $|Z(R)^*| \geq 2$. Then $diam(\Gamma(R)) \in \{1, 2, 3\}$ by [3, Theorem 2.3]. Hence, if $\Gamma(R) = AG(R)$, then $diam(\Gamma(R)) \in \{1, 2\}$ by Theorem 2.2. We recall the following results.

Lemma 3.1.

(1) [3, *the proof of Theorem* 2.8] *Let R be a reduced commutative ring that is not an integral domain. Then $\Gamma(R)$ is complete if and only if R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*
(2) [10, *Theorem* 2.6(3)] *Let R be a commutative ring. Then $diam(\Gamma(R)) = 2$ if and only if either (i) R is reduced with exactly two minimal primes and at least three nonzero zero divisors, or (ii) $Z(R)$ is an ideal whose square is not $\{0\}$ and each pair of distinct zero divisors has a nonzero annihilator.*

We first study the case when R is reduced.

Lemma 3.2. *Let R be a reduced commutative ring that is not an integral domain, and let $z \in Z(R)^*$. Then:*

(1) *$ann_R(z) = ann_R(z^n)$ for each positive integer $n \geq 2$;*
(2) *If $c + z \in Z(R)$ for some $c \in ann_R(z)\backslash\{0\}$, then $ann_R(z + c)$ is properly contained in $ann_R(z)$ (i.e., $ann_R(c + z) \subset ann_R(z)$). In particular, if $Z(R)$ is an ideal of R and $c \in ann_R(z)\backslash\{0\}$, then $ann_R(z + c)$ is properly contained in $ann_R(z)$.*

Proof. (1) Let $n \geq 2$. It is clear that $ann_R(z) \subseteq ann_R(z^n)$. Let $f \in ann_R(z^n)$. Since $fz^n = 0$ and R is reduced, we have $fz = 0$. Thus $ann_R(z^n) = ann_R(z)$.

(2) Let $c \in ann_R(z)\backslash\{0\}$, and suppose hat $c + z \in Z(R)$. Since $z^2 \neq 0$, we have $c + z \neq 0$, and hence $c + z \in Z(R)^*$. Since $c \in ann_R(z)$ and R is reduced, we have $c \notin ann_R(c + z)$. Hence $ann_R(c + z) \neq ann_R(z)$. Since $ann_R(c + z) \subset ann_R(z(c + z)) = ann_R(z^2)$ and $ann_R(z^2) = ann_R(z)$ by (1), it follows that $ann_R(c + z) \subset ann_R(z)$. □

Theorem 3.3. *Let R be a reduced commutative ring that is not an integral domain. Then the following statements are equivalent*:

(1) *$AG(R)$ is complete*;
(2) *$\Gamma(R)$ is complete* (*and hence $AG(R) = \Gamma(R)$*);
(3) *R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$*.

Proof. (1) ⇒ (2). Let $a \in Z(R)^*$. Suppose that $a^2 \neq a$. Since $ann_R(a^3) = ann_R(a)$ by Lemma 3.2(1) and $a^3 \neq 0$, we have $a - a^2$ is not an edge of $AG(R)$, a contradiction. Thus $a^2 = a$ for each $a \in Z(R)$. Let x, y be distinct elements in $Z(R)^*$. We show that $x - y$ is an edge of $\Gamma(R)$. Suppose that $xy \neq 0$. Since $x - y$ is an edge of $AG(R)$, we have $ann_R(xy) \neq ann_R(x)$, and thus $xy \neq x$. Since $x^2 = x$, we have $ann_R(x(xy)) = ann_R(x^2y) = ann_R(xy)$, and thus $x - xy$ is not an edge of $AG(R)$, a contradiction. Hence $xy = 0$ and $x - y$ is an edge of $\Gamma(R)$.

(2) ⇒ (3). It follows from Lemma 3.1(1).

(3) ⇒ (1). It is easily verified. □

Let R be a reduced commutative ring with $|Min(R)| \geq 2$. If $Z(R)$ is an ideal of R, then $Min(R)$ must be infinite, since $Z(R) = \cup\{Q \mid Q \in Min(R)\}$. For the construction of a reduced commutative ring R with infinitely many minimal prime ideals such that $Z(R)$ is an ideal of R, see [10, Section 5 (Examples)] and [1, Example 3.13].

Theorem 3.4. *Let R be a reduced commutative ring that is not an integral domain, and assume that $Z(R)$ is an ideal of R. Then $AG(R) \neq \Gamma(R)$ and $gr(AG(R)) = 3$.*

Proof. Let $z \in Z(R)^*$, $c \in ann_R(z)\backslash\{0\}$, and $m \in ann_R(c + z)\backslash\{0\}$. Then $m \in ann_R(c + z) \subset ann_R(z)$ by Lemma 3.2(2), and thus $mc = 0$. Since $c^2 \neq 0$, we have $m \neq c$, and hence $c + z \neq m + z$. Since $\{c, m\} \subseteq ann_R(z)$ and $z^2 \neq 0$, we have $c + z$ and $m + z$ are nonzero distinct elements of $Z(R)$. Since $(m + z)(c + z) = z^2 \neq 0$, we have $(c + z) - (m + z)$ is not an edge of $\Gamma(R)$. Since $c^2 \neq 0$ and $m^2 \neq 0$, it follows that $(c + m) \in ann_R(z^2)\backslash(ann_R(c + z) \cup ann_R(m + z))$, and thus $(c + z) - (m + z)$ is an edge of $AG(R)$. Since $(c + z) - (m + z)$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$, we have $AG(R) \neq \Gamma(R)$. Since R is reduced and $AG(R) \neq \Gamma(R)$, we have $gr(AG(R)) = 3$ by Theorem 2.6. □

Theorem 3.5. *Let R be a reduced commutative ring with $|Min(R)| \geq 3$ (possibly $Min(R)$ is infinite). Then $AG(R) \neq \Gamma(R)$ and $gr(AG(R)) = 3$.*

Proof. If $Z(R)$ is an ideal of R, then $AG(R) \neq \Gamma(R)$ by Theorem 3.4. Hence assume that $Z(R)$ is not an ideal of R. Since $|Min(R)| \geq 3$, we have $diam(\Gamma(R)) = 3$ by Lemma 3.1(2), and thus $AG(R) \neq \Gamma(R)$ by Theorem 2.2. Since R is reduced and $AG(R) \neq \Gamma(R)$, we have $gr(AG(R)) = 3$ by Theorem 2.6. □

Theorem 3.6. *Let R be a reduced commutative ring that is not an integral domain. Then $AG(R) = \Gamma(R)$ if and only if $|Min(R)| = 2$.*

Proof. Suppose that $AG(R) = \Gamma(R)$. Since R is a reduced commutative ring that is not an integral domain, $|Min(R)| = 2$ by Theorem 3.5. Conversely, suppose that $|Min(R)| = 2$. Let P_1, P_2 be the minimal prime ideals of R. Since R is reduced, we have $Z(R) = P_1 \cup P_2$ and $P_1 \cap P_2 = \{0\}$. Let $a, b \in Z(R)^*$. Assume that $a, b \in P_1$. Since $P_1 \cap P_2 = \{0\}$, neither $a \in P_2$ nor $b \in P_2$, and thus $ab \neq 0$. Since $P_1P_2 \subseteq P_1 \cap P_2 = \{0\}$, it follows that $ann_R(ab) = ann_R(a) = ann_R(b) = P_2$. Thus $a - b$ is not an edge of $AG(R)$. Similarly, if $a, b \in P_2$, then $a - b$ is not an edge of $AG(R)$. If $a \in P_1$ and $b \in P_2$, then $ab = 0$, and thus $a - b$ is an edge of $AG(R)$. Hence each edge of $AG(R)$ is an edge of $\Gamma(R)$, and therefore, $AG(R) = \Gamma(R)$. □

Theorem 3.7. *Let R be a reduced commutative ring. Then the following statements are equivalent*:

(1) $gr(AG(R)) = 4$;
(2) $AG(R) = \Gamma(R)$ *and* $gr(\Gamma(R)) = 4$;
(3) $gr(\Gamma(R)) = 4$;
(4) $T(R)$ *is ring-isomorphic to* $K_1 \times K_2$, *where each* K_i *is a field with* $|K_i| \geq 3$;
(5) $|Min(R)| = 2$ *and each minimal prime ideal of R has at least three distinct elements*;
(6) $\Gamma(R) = K^{m,n}$ *with* $m, n \geq 2$;
(7) $AG(R) = K^{m,n}$ *with* $m, n \geq 2$.

Proof. (1) ⇒ (2). Since $gr(AG(R)) = 4$, $AG(R) = \Gamma(R)$ by Theorem 2.6, and thus $gr(\Gamma(R)) = 4$. (2) ⇒ (3). No comments. (3) ⇔ (4) ⇔ (5) ⇔ (6) are clear by [2, Theorem 2.2]. (6) ⇒ (7). Since (6) implies $|Min(R)| = 2$ by [2, Theorem 2.2], we conclude that $AG(R) = \Gamma(R)$ by Theorem 3.6, and thus $gr(AG(R)) = gr(\Gamma(R)) = 4$. (7) ⇒ (1). This is clear since $AG(R)$ is a complete bipartite graph and $n, m \geq 2$. □

Theorem 3.8. *Let R be a reduced commutative ring that is not an integral domain. Then the following statements are equivalent*:

(1) $gr(AG(R)) = \infty$;
(2) $AG(R) = \Gamma(R)$ *and* $gr(AG(R)) = \infty$;
(3) $gr(\Gamma(R)) = \infty$;
(4) $T(R)$ *is ring-isomorphic to* $Z_2 \times K$, *where K is a field*;
(5) $|Min(R)| = 2$ *and at least one minimal prime ideal ideal of R has exactly two distinct elements*;
(6) $\Gamma(R) = K^{1,n}$ *for some* $n \geq 1$;
(7) $AG(R) = K^{1,n}$ *for some* $n \geq 1$.

Proof. (1) ⇒ (2). Since $gr(AG(R)) = \infty$, $AG(R) = \Gamma(R)$ by Theorem 2.6, and thus $gr(\Gamma(R)) = \infty$. (2) ⇒ (3). No comments. (3) ⇔ (4) ⇔ (5) ⇔ (6) are clear by

[2, Theorem 2.4]. (6) $\Rightarrow$ (7). Since (6) implies $|Min(R)| = 2$ by [2, Theorem 2.4], we conclude that $AG(R) = \Gamma(R)$ by Theorem 3.6, and thus $gr(AG(R)) = gr(\Gamma(R)) = \infty$. (7) $\Rightarrow$ (1). It is clear. □

In view of Theorem 3.7 and Theorem 3.8, we have the following result.

Corollary 3.9. *Let R be a reduced commutative ring. Then $AG(R) = \Gamma(R)$ if and only if $gr(AG(R)) = gr(\Gamma(R)) \in \{4, \infty\}$.*

For the remainder of this section, we study the case when R is nonreduced.

Theorem 3.10. *Let R be a nonreduced commutative ring with $|Nil(R)^*| \geq 2$ and let $AG_N(R)$ be the (induced) subgraph of $AG(R)$ with vertices $Nil(R)^*$. Then $AG_N(R)$ is complete.*

Proof. Suppose there are nonzero distinct elements $a, b \in Nil(R)$ such that $ab \neq 0$. Assume that $ann_R(ab) = ann_R(a) \cup ann_R(b)$. Hence $ann_R(ab) = ann_R(a)$ or $ann_R(ab) = ann_R(b)$. Without lost of generality, we may assume that $ann_R(ab) = ann_R(a)$. Let n be the least positive integer such that $b^n = 0$. Suppose that $ab^k \neq 0$ for each k, $1 \leq k < n$. Then $b^{n-1} \in ann_R(ab) \backslash ann_R(a)$, a contradiction. Hence assume that k, $1 \leq k < n$ is the least positive integer such that $ab^k = 0$. Since $ab \neq 0$, $1 < k < n$. Hence $b^{k-1} \in ann_R(ab) \backslash ann_R(a)$, a contradiction. Thus $a - b$ is an edge of $AG_N(R)$. □

In view of Theorem 3.10, we have the following result.

Corollary 3.11. *Let R be a nonreduced quasi-local commutative ring with maximal ideal $Nil(R)$ such that $|Nil(R)^*| \geq 2$. Then $AG(R)$ is complete. In particular, $AG(\mathbb{Z}_{2^n})$ is complete for each $n \geq 3$ and if $q > 2$ is a positive prime number of $\mathbb{Z}$, then $AG(\mathbb{Z}_{q^n})$ is complete for each $n \geq 2$.*

The following is an example of a quasi-local commutative ring R with maximal ideal $Nil(R)$ such that $w^2 = 0$ for each $w \in Nil(R)$, $diam(\Gamma(R)) = 2$, $diam(AG(R)) = 1$, and $gr(AG(R)) = gr(\Gamma(R)) = 3$.

Example 3.12. Let $R = \mathbb{Z}_2[X, Y]/(X^2, Y^2)$, $x = X + (X^2, Y^2) \in R$, and $y = Y + (X^2, Y^2) \in R$. Then R is a quasi-local commutative ring with maximal ideal $Nil(R) = (x, y)R$. It is clear that $w^2 = 0$ for each $w \in Nil(R)$ and $diam(AG(R)) = 1$ by Corollary 3.11. Since $Nil(R)^2 \neq \{0\}$ and $xyNil(R) = \{0\}$, we have $diam(\Gamma(R)) = 2$ by Lemma 3.1(2). Since $x - xy - (xy + x) - x$ is a cycle of length three in $\Gamma(R)$, we have $gr(AG(R)) = gr(\Gamma(R)) = 3$.

Theorem 3.13. *Let R be a nonreduced commutative ring with $|Nil(R)^*| \geq 2$, and let $\Gamma_N(R)$ be the induced subgraph of $\Gamma(R)$ with vertices $Nil(R)^*$. Then $\Gamma_N(R)$ is complete if and only if $Nil(R)^2 = \{0\}$.*

Proof. If $Nil(R)^2 = \{0\}$, then it is clear that $\Gamma_N(R)$ is complete. Hence assume that $\Gamma_N(R)$ is complete. We need only show that $w^2 = 0$ for each $w \in Nil(R)^*$.

Let $w \in Nil(R)^*$ and assume that $w^2 \neq 0$. Let n be the least positive integer such that $w^n = 0$. Then $n \geq 3$. Thus $w, w^{n-1} + w$ are distinct elements of $Nil(R)^*$. Since $w(w^{n-1} + w) = 0$ and $w^n = 0$, we have $w^2 = 0$, a contradiction. Thus $w^2 = 0$ for each $w \in Nil(R)$. □

Theorem 3.14. *Let R be a nonreduced commutative ring, and suppose that $Nil(R)^2 \neq \{0\}$. Then $AG(R) \neq \Gamma(R)$ and $gr(AG(R)) = 3$.*

Proof. Since $Nil(R)^2 \neq \{0\}$, $AG(R) \neq \Gamma(R)$ by Theorem 3.10 and Theorem 3.13. Thus $gr(AG(R)) \in \{3, 4\}$ by Corollary 2.11. Let $F = \mathbb{Z}_2 \times B$, where B is $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Since $Nil(F)^2 = \{0\}$ and $Nil(F) \neq \{0\}$, we have $gr(AG(R)) \neq 4$ by Theorem 2.9. Thus $gr(AG(R)) = 3$. □

Theorem 3.15. *Let R be a nonreduced commutative ring such that $Z(R)$ is not an ideal of R. Then $AG(R) \neq \Gamma(R)$.*

Proof. Since R is nonreduced and $Z(R)$ is not an ideal of R, $diam(\Gamma(R)) = 3$ by [10, Corollary 2.5]. Hence $AG(R) \neq \Gamma(R)$ by Theorem 2.2. □

Theorem 3.16. *Let R be a nonreduced commutative ring. Then the following statements are equivalent:*

(1) *$gr(AG(R)) = 4$;*
(2) *$AG(R) \neq \Gamma(R)$ and $gr(AG(R)) = 4$;*
(3) *R is ring-isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$;*
(4) *$\Gamma(R) = \overline{K}^{1,3}$;*
(5) *$AG(R) = K^{2,3}$.*

Proof. (1) ⇒ (2). Suppose $AG(R) = \Gamma(R)$. Then $gr(\Gamma(R)) = 4$, and R is ring-isomorphic to $D \times B$, where D is an integral domain with $|D| \geq 3$ and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by [2, Theorem 2.3]. Assume that R is ring-isomorphic to $D \times \mathbb{Z}_4$. Since $|D| \geq 3$, there is an $a \in D\backslash\{0, 1\}$. Let $x = (0, 1)$, $y = (1, 2)$, $w = (a, 2) \in R$. Then x, y, w are distinct elements in $Z(R)^*$, $w(xy) = (0, 0)$, $wx \neq (0, 0)$, and $wy \neq (0, 0)$. Thus $x - w - y - x$ is a cycle of length three in $AG(R)$ by Lemma 2.3, a contradiction. Similarly, assume that R is ring-isomorphic to $D \times \mathbb{Z}_2[X]/(X^2)$. Again, since $|D| \geq 3$, there is an $a \in D\backslash\{0, 1\}$. Let $x = X + (X^2) \in \mathbb{Z}_2[X]/(X^2)$. Then it is easily verified that $(0, 1) - (a, x) - (1, x) - (0, 1)$ is a cycle of length three in $AG(R)$, a contradiction. Thus $AG(R) \neq \Gamma(R)$. (2) ⇒ (3). It is clear by Theorem 2.9. (3) ⇔ (4). It is clear by [2, Theorem 2.5]. (4) ⇒ (5). Since (4) implies (3) by [2, Theorem 2.5], it is easily verified that the annihilator graph of the two rings in (3) is $K^{2,3}$. (4) ⇒ (5). Since $AG(R)$ is a $K^{2,3}$, it is clear that $gr(AG(R)) = 4$. □

We observe that $gr(\Gamma(\mathbb{Z}_8)) = \infty$, but $gr(AG(\mathbb{Z}_8)) = 3$. We have the following result.

Theorem 3.17. *Let R be a commutative ring such that $AG(R) \neq \Gamma(R)$. Then the following statements are equivalent:*

(1) *$\Gamma(R)$ is a star graph;*

(2) $\Gamma(R) = K^{1,2}$;
(3) $AG(R) = K^3$.

Proof. (1) $\Rightarrow$ (2). Since $gr(\Gamma(R)) = \infty$ and $\Gamma(R) \neq AG(R)$, we have R is non-reduced by Corollary 3.9 and $|Z(R)^*| \geq 3$. Since $\Gamma(R))$ is a star graph, there are two sets A, B such that $Z(R)^* = A \cup B$ with $|A| = 1$, $A \cap B = \emptyset$, $AB = \{0\}$, and $b_1 b_2 \neq 0$ for every $b_1, b_2 \in B$. Since $|A| = 1$, we may assume that $A = \{w\}$ for some $w \in Z(R)^*$. Since each edge of $\Gamma(R)$ is an edge of $AG(R)$ and $AG(R) \neq \Gamma(R)$, there are some $x, y \in B$ such that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$. Since $ann_R(c) = w$ for each $c \in B$ and $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$, we have $ann_R(xy) \neq w$. Thus $ann_R(xy) = B$ and $xy = w$. Since $A = \{xy\}$ and $AB = \{0\}$, we have $x(xy) = x^2 y = 0$ and $y(xy) = y^2 x = 0$. We show that $B = \{x, y\}$, and hence $|B| = 2$. Thus assume there is a $c \in B$ such that $c \neq x$ and $c \neq y$. Then $wc = xyc = 0$. We show that $(xc + xy) \neq x$ and $(xc + xy) \neq xy$ (note that $xy = w$). Suppose that $(xc + xy) = x$. Then $y(xc + xy) = yx$. But $y(xc + xy) = yxc + xy^2 = 0 + 0 = 0$ and $xy \neq 0$, a contradiction. Hence $x \neq (xc + xy)$. Since $x, c \in B$, we have $xc \neq 0$ and thus $(xc + xy) \neq xy$. Thus $x, (xc + xy), xy$ are distinct elements of $Z(R)^*$. Since $x^2 y = 0$ and $y \in B$, either $x^2 = 0$ or $x^2 = xy$ or $x^2 = y$. Suppose that $x^2 = y$. Since $xy = w \neq 0$, we have $xy = x(x^2) = x^3 = w \neq 0$. Since $x^2 y = 0$, we have $x^4 = 0$. Since $x^4 = 0$ and $x^3 \neq 0$, we have $x^2, x^3, x^2 + x^3$ are distinct elements of $Z(R)^*$, and thus $x^2 - x^3 - (x^2 + x^3) - x^2$ is a cycle of length three in $\Gamma(R)$, a contradiction. Hence, we assume that either $x^2 = 0$ or $x^2 = xy = w$. In both cases, we have $x^2 c = 0$. Since $x, (xc + xy), xy$ are distinct elements of $Z(R)^*$ and $xy^2 = yx^2 = x^2 c = 0$, we have $x - (xc + xy) - xy - x$ is a cycle of length three in $\Gamma(R)$, a contradiction. Thus $B = \{x, y\}$ and $|B| = 2$. Hence $\Gamma(R) = K^{1,2}$. (2) $\Rightarrow$ (3). Since each edge of $\Gamma(R)$ is an edge of $AG(R)$ and $\Gamma(R) \neq AG(R)$ and $\Gamma(R) = K^{1,2}$, it is clear that $AG(R)$ must be K^3. (3) $\Rightarrow$ (1). Since $|Z(R)^*| = 3$ and $\Gamma(R)$ is connected and $AG(R) \neq \Gamma(R)$, exactly one edge of $AG(R)$ is not an edge of $\Gamma(R)$. Thus $\Gamma(R)$ is a star graph. □

Theorem 3.18. *Let R be a non-reduced commutative ring with $|Z(R)^*| \geq 2$. Then the following statements are equivalent*:

(1) *$AG(R)$ is a star graph*;
(2) *$gr(AG(R)) = \infty$*;
(3) *$AG(R) = \Gamma(R)$ and $gr(\Gamma(R)) = \infty$*;
(4) *$Nil(R)$ is a prime ideal of R and either $Z(R) = Nil(R) = \{0, -w, w\}$ $(w \neq -w)$ for some nonzero $w \in R$ or $Z(R) \neq Nil(R)$ and $Nil(R) = \{0, w\}$ for some nonzero $w \in R$ (and hence $wZ(R) = \{0\}$)*;
(5) *Either $AG(R) = K^{1,1}$ or $AG(R) = K^{1,\infty}$*;
(6) *Either $\Gamma(R) = K^{1,1}$ or $\Gamma(R) = K^{1,\infty}$.*

Proof. (1) $\Rightarrow$ (2). It is clear by the definition of the star graph. (2) $\Rightarrow$ (3). Since $gr(AG(R)) = \infty$, $AG(R) = \Gamma(R)$ by Corollary 2.11, and thus $gr(\Gamma(R)) = \infty$. (3) $\Rightarrow$ (4). Suppose that $|Nil(R)^*| \geq 3$. Since $AG_N(R)$ is complete by Theorem 3.10 and $|Nil(R)^*| \geq 3$, we have $gr(AG(R)) = gr(\Gamma(R)) = 3$, a contradiction. Thus $|Nil(R)^*| \in \{1, 2\}$. Suppose $|Nil(R)^*| = 2$. Then $Nil(R) = \{0, w, -w\}$ $(w \neq -w)$ for some nonzero $w \in R$. We show $Z(R) = Nil(R)$. Assume there is a $k \in Z(R) \setminus Nil(R)$. Suppose that $wk = 0$. Since $Nil(R)^2 = \{0\}$, $w - k - (-w) - w$ is a cycle of length three in $\Gamma(R)$, a contradiction. Thus assume that $wk \neq 0$. Hence there is an $f \in Z(R)^* \backslash \{w, -w, k\}$, such that $w - f - z$ is a path of length two in $\Gamma(R)$ by Theorem 2.2 (note that we are assuming that $AG(R) = \Gamma(R)$). Thus $w - f - (-w) - w$ is a cycle of length three in $\Gamma(R)$, a contradiction. Hence if $|Nil(R)^*| = 2$, then $Z(R) = Nil(R)$. Thus assume that $Nil(R) = \{0, w\}$ for some nonzero $w \in R$. We show $Nil(R)$ is a prime ideal of R. Since $gr(AG(R)) = gr(\Gamma(R)) = \infty$, we have $AG(R) = \Gamma(R)$ is a star graph by [2, Theorem 2.5] and Theorem 3.16. Since $|Z(R)^*| \geq 2$ by hypothesis and $|Nil(R)^*| = 1$, we have $Z(R) \neq Nil(R)$. Let $c \in Z(R)^* \backslash Nil(R)^*$. We show $wc = 0$. Suppose that $wc \neq 0$. Since $|Nil(R)^*| = 1$ and $wc \neq 0$, we have $wc = w$. Thus $w(c-1) = 0$. Since $w + 1 \in U(R)$ and $c \notin U(R)$, we have $c - 1 \neq w$. Since $\Gamma(R)$ is a star graph and $w(c-1) = 0$ and $wc \neq 0$, we have $(c-1)j = 0$ for each $j \in Z(R)^* \backslash \{c-1\}$. In particular, $(c-1)[(c-1)+w] = 0$, and therefore $w - (c-1) - (c-1+w) - w$ is a cycle of length three in $\Gamma(R)$, a contradiction. Hence $wc = 0$. Since $wZ(R) = \{0\}$ and $\Gamma(R)$ is a star graph, we have $Nil(R) = \{0, w\}$ is a prime ideal of R. (4) $\Rightarrow$ (5). Suppose that $Nil(R)$ is a prime ideal of R. If $Z(R) = Nil(R)$ and $|Nil(R)^*| = 2$, then $AG(R) = K^{1,1}$. Hence, assume that $Nil(R) = \{0, w\}$ for some nonzero $w \in R$. We show that $Z(R)$ is an infinite set. Let $c \in Z(R) \backslash Nil(R)$ and let $n > m \geq 1$. We show that $c^m \neq c^n$. Suppose that $c^m = c^n$. Then $c^m(1 - c^{n-m}) = 0$. Since $Nil(R) = \{0, w\}$ is a prime ideal of R, we have $(1 - c^{n-m}) = w$. Since $1 - w \in U(R)$, we have $1 - w = c^{n-m} \in U(R)$, a contradiction. Thus $c^m \neq c^n$, and hence $Z(R)$ is an infinite set. Since $Nil(R) = \{0, w\}$ is a prime ideal of R and $wZ(R) = \{0\}$, we have $AG(R) = K^{1,\infty}$. (5) $\Rightarrow$ (6). It is clear. (6) $\Rightarrow$ (1). Since $\Gamma(R)$ is a star graph and $\Gamma(R) \neq K^{1,2}$, we have $AG(R) = \Gamma(R)$ by Theorem 3.17, and thus $gr(AG(R)) = \infty$. □

Corollary 3.19 ([3, Theorem 2.13], [2, Remark 2.6(a)], and [4, Theorem 3.9])**.** *Let R be a nonreduced commutative ring with $|Z(R)^*| \geq 2$. Then $\Gamma(R)$ is a star graph if and only if $\Gamma(R) = K^{1,1}$, $\Gamma(R) = K^{1,2}$, or $\Gamma(R) = K^{1,\infty}$.*

Proof. The proof is a direct implication of Theorems 3.17 and 3.18. □

In the following example, we construct two nonreduced commutative rings say R_1 and R_2, where $AG(R_1) = K^{1,1}$ and $AG(R_2) = K^{1,\infty}$.

Example 3.20.

(1) Let $R_1 = \mathbb{Z}_3[X]/(X^2)$, and let $x = X + (X^2) \in R_1$. Then $Z(R_1) = Nil(R_1) = \{0, -x, x\}$ and $AG(R_1) = \Gamma(R_1) = K^{1,1}$. Also note that $AG(\mathbb{Z}_9) = \Gamma(\mathbb{Z}_9) = K^{1,1}$.
(2) Let $R_2 = \mathbb{Z}_2[X, Y]/(XY, X^2)$. Then let $x = X + (XY + X^2)$ and $y = Y + (XY + X^2) \in R_2$. Then $Z(R_2) = (x, y)R_2$, $Nil(R_2) = \{0, x\}$, and $Z(R_2) \neq Nil(R_2)$. It is clear that $AG(R_2) = \Gamma(R_2) = K^{1,\infty}$.

Remark 3.21. Let R be a nonreduced commutative ring. In view of Theorem 3.15, Theorem 3.16, and Theorem 3.18, if $AG(R) = \Gamma(R)$, then $Z(R)$ is an ideal of R and $gr(AG(R)) = gr(\Gamma(R)) \in \{3, \infty\}$. The converse is true if $gr(AG(R) = gr(\Gamma(R)) = \infty$

(see Theorems 3.15 and 3.18). However, if $Z(R)$ is an ideal of R and $gr(AG(R)) = gr(\Gamma(R)) = 3$, then it is possible to have all the following cases:

(1) It is possible to have a commutative ring R such that $Z(R)$ is an ideal of R, $Z(R) \neq Nil(R)$, $AG(R) = \Gamma(R)$, and $gr(AG(R)) = 3$. See Example 3.22;
(2) It is possible to have a commutative ring R such that $Z(R)$ is an ideal of R, $Z(R) \neq Nil(R)$, $Nil(R)^2 = \{0\}$, $AG(R) \neq \Gamma(R)$, $diam(AG(R)) = diam(\Gamma(R)) = 2$, and $gr(AG(R)) = gr(\Gamma(R)) = 3$. See Example 3.23.
(3) It is possible to have a commutative ring R such that $Z(R)$ is an ideal of R, $Z(R) \neq Nil(R)$, $Nil(R)^2 = \{0\}$, $AG(R)$ is a complete graph (i.e., $diam(AG(R)) = 1$), $AG(R) \neq \Gamma(R)$, $diam(\Gamma(R)) = 2$, and $gr(AG(R)) = gr(\Gamma(R)) = 3$. See Theorem 3.24.

Example 3.22. Let $D = \mathbb{Z}_2[X, Y, W]$, $I = (X^2, Y^2, XY, XW)D$ be an ideal of D, and let $R = D/I$. Then let $x = X + I$, $y = Y + I$, and $w = W + I$ be elements of R. Then $Nil(R) = (x, y)R$ and $Z(R) = (x, y, w)R$ is an ideal of R. By construction, we have $Nil(R)^2 = \{0\}$, $AG(R) = \Gamma(R)$, $diam(AG(R)) = diam(\Gamma(R)) = 2$, and $gr(AG(R)) = gr(\Gamma(R)) = 3$ (for example, $x - (x + y) - y - x$ is a cycle of length three).

Example 3.23. Let $D = \mathbb{Z}_2[X, Y, W]$, $I = (X^2, Y^2, XY, XW, YW^3)D$ be an ideal of D, and let $R = D/I$. Then let $x = X + I, y = Y + I$, and $w = W + I$ be elements of R. Then $Nil(R) = (x, y)R$ and $Z(R) = (x, y, w)R$ is an ideal of R. By construction, $Nil(R)^2 = \{0\}$, $diam(AG(R)) = diam(\Gamma(R)) = 2$, $gr(AG(R)) = gr(\Gamma(R)) = 3$. However, since $w^3 \neq 0$ and $y \in ann_R(w^3) \backslash (ann_R(w) \cup ann_R(w^2))$, we have $w - w^2$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$, and hence $AG(R) \neq \Gamma(R)$.

Given a commutative ring R and an R-module M, the *idealization* of M is the ring $R(+)M = R \times M$ with addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$ for all $r, s \in R$ and $m, n \in M$. Note that $\{0\}(+)M \subseteq Nil(R(+)M)$ since $(\{0\}(+)M)^2 = \{(0, 0)\}$. We have the following result.

Theorem 3.24. *Let D be a principal ideal domain that is not a field with quotient field K (for example, let $D = \mathbb{Z}$ or $D = F[X]$ for some field F), and let $Q = (p)$ be a nonzero prime ideal of D for some prime (irreducible) element $p \in D$. Set $M = K/D_Q$ and $R = D(+)M$. Then $Z(R) \neq Nil(R)$, $AG(R)$ is a complete graph, $AG(R) \neq \Gamma(R)$, and $gr(AG(R)) = gr(\Gamma(R)) = 3$.*

Proof. By construction of R, $Z(R) = Q(+)M$, $Nil(R) = \{0\}(+)M$, and $Nil(R)^2 = \{(0, 0)\}$. Let x, y be distinct elements of $Z(R)^*$, and suppose that $xy \neq 0$. Since $Nil(R)^2 = \{(0, 0)\}$, to show that $AG(R)$ is complete, we consider two cases. Case I: assume $x \in Nil(R)^*$ and $y \in Z(R) \backslash Nil(R)$. Then $x = (0, \frac{a}{cp^m} + D_Q)$ for some nonzero $a \in D$, $c \in D \backslash Q$, and some positive integer $m \geq 1$ such that $gcd(a, cp^m) = 1$, and $y = (hp^n, f)$ for some positive integer $n \geq 1$, a nonzero $h \in D$, and $f \in M$. Since $xy \neq 0$, we have $n < m$. Hence $xy = (0, \frac{ha}{cp^{m-n}} + D_Q) \in Nil(R)^*$. Since $(p^{m-n}, 0) \in ann_R(xy) \backslash (ann_R(x) \cup ann_R(y))$, we have $x - y$ is an edge of $AG(R)$. Case II: assume that $x, y \in Z(R)^* \backslash Nil(R)^*$. Then $x = (dp^u, g)$ and $y = (vp^r, w)$ for some positive integers $u, r \geq 1$, nonzero $d, v \in D\backslash Q$, and $g, w \in M$. Hence $xy = (dvp^{u+r}, dp^u w + vp^r g)$. Since $(0, \frac{1}{p^{u+r}} + D_Q) \in ann_R(xy)\backslash(ann_R(x) \cup ann_R(y))$, we have $x - y$ is an edge of $AG(R)$. Since $(0, \frac{1}{p} + D_Q) - (0, \frac{1}{p^2} + D_Q) - (0, \frac{1}{p^3} + D_Q) - (0, \frac{1}{p} + D_Q)$ is a cycle of length three in $\Gamma(R)$, we have $gr(AG(R)) = gr(\Gamma(R)) = 3$. □

The following example shows that the hypothesis "Q is principal" in the above theorem is crucial.

Example 3.25. Let $D = \mathbb{Z}[X]$ with quotient field K and $Q = (2, X)D$. Then Q is a nonprincipal prime ideal of D. Set $M = K/D_Q$ and $R = D(+)M$. Then $Z(R) = Q(+)M$, $Nil(R) = \{0\}(+)M$, and $Nil(R)^2 = \{(0, 0)\}$. Let $a = (2, 0)$ and $b = (0, \frac{1}{X} + D_Q)$. Then $ab = (0, \frac{2}{X} + D_Q) \in Nil(R)^*$. Since $ann_R(ab) = ann_R(b)$, we have $a - b$ is not an edge of $AG(R)$. Thus $AG(R)$ is not a complete graph.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to the referee for his (her) great effort in proofreading the manuscript and for the several comments.

REFERENCES

[1] Anderson, D. F. (2008). On the diameter and girth of a zero-divisor graph, II. *Houston J. Math.* 34:361–371.
[2] Anderson, D. F., Mulay, S. B. (2007). On the diameter and girth of a zero-divisor graph. *J. Pure Appl. Algebra* 210(2):543–550.
[3] Anderson, D. F., Livingston, P. S. (1999). The zero-divisor graph of a commutative ring. *J. Algebra* 217:434–447.
[4] Anderson, D. F., Levy, R., Shapiro, J. (2003). Zero-divisor graphs, von Neumann regular rings, and boolean algebras. *JPAA* 180:221–241.
[5] Anderson, D. F., Axtell, M. C., Stickles, J. A. Jr. (2011). Zero-divisor graphs in commutative rings. In: Fontana, M., Kabbaj, S.-E., Olberding, B., Swanson, I., eds. *Commutative Algebra, Noetherian and Non-Noetherian Perspectives*. New York: Springer-Verlag, pp. 23–45.
[6] Beck, I. (1988). Coloring of commutative rings. *J. Algebra* 116:208–226.
[7] Bollaboás, B. (1998). *Modern Graph Theory*. New York: Springer-Verlag.
[8] Hung-Jen Chiang-Hsieh (2008). Classification of rings with projective zero-divisor graphs. *J. Algebra* 319:2789–2802.
[9] Huckaba, J. A. (1988). *Commutative Rings with Zero Divisors*. New York/Basel: Marcel Dekker.
[10] Lucas, T. G. (2006). The diameter of a zero-divisor graph. *J. Algebra* 301:3533–3558.
[11] Hsin-Ju Wang (2006). Zero-divisor graphs of genus one. *J. Algebra* 304:666–678.
[12] Wickham, C. (2009). Rings whose zero-divisor graphs have positive genus. *J. Algebra* 321:377–383.
[13] Akbari, S., Maimani, H. R., Yassemi, S. (2003). When a zero-divisor graph is planar or a complete r-partite graph. *J. Algebra* 270:169–180.
[14] Maimani, H. R., Pournaki, M. R., Tehranian, A., Yassemi, S. (2011). Graphs attached to rings revisited. *Arab. J. Sci. Eng.* 36:997–1012.