Name——————————————————, ID ——————

HW III

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QUESTION 1. (a) Find two positive integers a, b such that $a^2 + b^2 = 41$

HINT: You may use this technique if you choose. First note that $41 = 4(10) + 1$ **is prime. By try and error find two integers, x, y such that** $x^2 + y = 10$ **and** $4y + 1 = b^2$ **for some integer b. Let x = 2 and y = 6. Then** $41 = 4(10) + 1 = 4(2^2 + 6) + 1 = 4^2 + 4(6) + 1 = 4^2 + 5^2$**.**

(b) Find two positive integers a, b such that $a^2 + b^2 = 89$

HINT: First note that $89 = 4(22) + 1$ **is prime. By try and error, see my comments in (a). Let x = 4 and y = 6. Then** $89 = 4(22) + 1 = 4(4^2 + 6) + 1 = 8^2 + 4(6) + 1 = 8^2 + 5^2$**.**

(c) Find two positive integers a, b such that $a^2 + b^2 = 23$

HINT: It is impossible since for every relatively prime positive integers, a, b, $a^2 + b^2 (mod\ 4)$ **must be 1, see HW2. However,** $23\ (mod\ \ 4) = 3$

(d) Is $33 = a^2 + b^2$ for some integers a, b?

HINT: No, note 33 is not prime, so this does not contradict Q1. $33 = 4(8) + 1$ **, try and error, not equal** $a^2 + b^2$**.**

(e) Is $45 = a^2 + b^2$ for some integers a, b?

HINT: yes, note 45 is not prime, $45 = 4(11) + 1$ **, try and error, see my comments in (a),** $45 = 4(11) + 1 = 4(3^2 + 2) + 1 = 6^2 + 8 + 1 = 6^2 + 3^2$

(f) Is $49 = a^2 + b^2$ for some integers a, b?

HINT: No, note 49 is not prime, so this does not contradict Q1. $49 = 4(12) + 1$ **, try and error, not equal** $a^2 + b^2$**.**

QUESTION 2. Let $n = 100(7^3)(5^4)(2^3)$. Let D be the set of all divisors of n.

(i) Find $|D|$. (note that the prime factorization of n is $(2^5)(5^6)(7^3)$, see class notes.)

(ii) Find $\sigma(n)$, i.e., the sum of all divisors of n. See class notes.

(iii) Let $F = \{d \in D$, such that, $100 \mid d\}$. Find $|F|$. Find $\sum_{f \in F} f$.

Hint. Let $f \in F$**. Then** f **has the form** $2^i 5^k 7^j$**, where** $2 \leq i \leq 5$**,** $2 \leq k \leq 6$**,** $0 \leq j \leq 3$ **and see class notes**

(iv) Let $F = \{d \in D$, such that, $10 = gcd(100, d)\}$. Find $|D|$. Find $\sum_{f \in F} f$.

Hint. Let $f \in F$**. Then** f **has the form** $(2)(5)7^j$**, where** $0 \leq j \leq 3$ **and see class notes**

(v) Let $F = \{d \in D$, such that, $70 = gcd(350, d)\}$. Find $|F|$. Find $\sum_{f \in F} f$.

Hint. Let $f \in F$**. Then** f **has the form** $2^i(5)7^j$**, where** $1 \leq i \leq 6$**,** $1 \leq j \leq 3$ **and see class notes**

(vi) Let $F = \{d \in D$, such that, $25 = gcd(100, d)\}$. Find $|F|$. Find $\sum_{f \in F} f$.

Hint. Let $f \in F$**. Then** f **has the form** $5^i 7^j$**, where** $2 \leq i \leq 6$**,** $0 \leq j \leq 3$ **and see class notes**

QUESTION 3. Assume that a, b, c are positive integers such that $a \mid bc$. Assume that $gcd(a, b) = 1$. Prove that $a \mid c$.

Hint: We know that (*) $1 = an + bm$ **for some integers** $n.m$**. Multiply (*) by c, we get (**)** $c = can + cbm$**. By staring at (**), a is a factor of the right-hand side of (**), i.e.,** $a|(can + cbm)$**. Thus** a **is a factor of** c**.**

QUESTION 4. Let $n = d_1d_2\cdots d_m$, $m \geq 5$, where $1 \leq d_1 \leq 9$, and $0 \leq d_i \leq 9$ for every $2 \leq i \leq m$. Assume that $101 \mid n$. Prove that $101 \mid (d_1 \cdots d_{(m-2)} - d_{(m-1)}d_m)$. (see class notes, but multiply with 100)

QUESTION 5. Prove the converse of Q5.

Hint: Assume that $101 \mid (d_1 \cdots d_{(m-2)} - d_{(m-1)}d_m)$**. It is clear that** $101 \mid 100(d_1 \cdots d_{(m-2)} - d_{(m-1)}d_m)$**. We show** $101 \mid n$**. Now,** $100(d_1 \cdots d_{(m-2)} - d_{(m-1)}d_m) = d_1 \cdots d_{(m-2)}00 - 100d_{(m-1)}d_m = d_1 \cdots d_{(m-2)}00 + \underline{d_{(m-1)}d_m - d_{(m-1)}d_m} - 100d_{(m-1)}d_m = n - 101d_{(m-1)}d_m$**. Hence** $101 \mid (n - 101d_{(m-1)}d_m)$ **. By staring, since** $101 \mid 101d_{(m-1)}d_m$**, we conclude that** $101 \mid n$**.**

QUESTION 6. Show the work. Use the algorithm as in Q5 and Q6.

(i) Is 5,656 divisible by 101?

(ii) Is 12,423 divisible by 101?

(iii) Is 54,134 divisible by 101?

QUESTION 7. Let $n = d_1d_2\cdots d_m$, $m \geq 4$, where $1 \leq d_1 \leq 9$, and $0 \leq d_i \leq 9$ for every $2 \leq i \leq m$. Assume that $27 \mid n$. Prove that $27 \mid (d_1 \cdots d_{(m-1)} - 8d_m)$. (see class notes)

QUESTION 8. Prove the converse of Q8.(see the proof of Q6, maybe you need to multiply with 10)

QUESTION 9. Let $n = d_1d_2\cdots d_m$, $m \geq 4$, where $1 \leq d_1 \leq 9$, and $0 \leq d_i \leq 9$ for every $2 \leq i \leq m$. Assume that for some positive integer $k > 1$, $k \mid n$ if and only if $k \mid (d_1 \cdots d_{(m-1)} - 8d_m)$. Find all possibilities of k. Think, it is not hard.

QUESTION 10. Show the work. Use the algorithm as in Q8 and Q9.

(i) Is 2, 862 divisible by 27?

(ii) Is 50,301 divisible by 27?

(iii) Is 16, 252 divisible by 27?

QUESTION 11. Find the largest positive integer n such that $(n + 8) \mid (n^3 + 80)$. (Answer: n = 424, see class notes)

QUESTION 12. Find the largest positive integer n such that $(n+9) \mid (n^2+90)$.(Answer: n = 162, see class notes)

QUESTION 13. Find the largest positive integer n such that $(n+29) \mid (n^2+n+23)$.(Answer: n = 806, see class notes)

QUESTION 14. Find the largest positive integer n such that $(n + 21) \mid (3n^4 + 5n + 10)$.(Answer: n = 583327, see class notes)

QUESTION 15. Find the largest positive integer n such that $(n + 37) \mid (n + 37)^{12} + 547)$.(Answer: n = 510, see class notes)

QUESTION 16. Consider $U(26)$.

(a) can we generate U(26) by one element? If yes find a generator.
HINT: Yes/ see class notes, 26 = 2(13). $|U(26)| = \phi(26) = 12$**. Since** $4 \mid 12$**, find a in U(26) such that** $a^6 = -1 = 25\ (mod\ 26)$ **and** $a^4 \neq 1$**, 5 will not work since** $a^6 = 25\ (mod\ 26)$**, but** $5^4 = 1 (mod\ 26)$**, a = 7 will generate U(26) .**

(b) Find $4 - R(U(26))$. Note that $4 - R(U(26)) = C(3) \subset U(26) = \{7^4, (7^4)^2, (7^4)^3 = 1\} = \{9, 3, 1\}$.

(c) Solve for x over U(26), $x^4 = 3$.
HINT: Find $C(4)$ **in** $U(26)$**,** $C(4) = \{7^3, (7^3)^2, (7^3)^3, (7^3)^4 = 1\} = \{5, 25, 21, 1\}$**. Now from (a), it is clear that** $x = 7^2 = 23$ **is a solution for** $x^4 = 3$**. Hence the solution set in U(26) is** $23C(4) = \{11, 3, 15, 23\}$

(d) Find all integers over Z, say x, such that $gcd(x, 26) = 1$ and $x^4\ (mod\ 26) = 3$. No comments, use (c).

QUESTION 17. Consider $U(49)$.

(a) can we generate U(49) by one element? If yes find a generator.
HINT: Yes/ see class notes, $49 = 7^2$**).** $|U(49)| = \phi(49) = 42$**. Since** $4 \nmid 42$**, find a in U(49) such that** $a^{21} = -1 = 48 \ (mod\, 49)$**, a = 3 will generate U(49) .**

(b) Find $6 - R(U(26))$. Note that $6 - R(U(49)) = C(7) \subset U(49)$. See Q17

(c) Solve for x over U(49), $x^6 = 43$. See Q17

(d) Find all integers over Z, say x, such that $gcd(x, 49) = 1$ and $x^6 \ (mod\, 49) = 43$. No comments, use (c).

(e) Is $18 \in 7 - R(U(49))$? (yes if $18^6 = 1$, see HW2). Is $31 \in 7 - R(U(49))$?. Is $9 \in 21 - R(U(49))$?

QUESTION 18. Prove that $(p-1)! \ (mod \ p) = p - 1$ for every odd prime positive integer. (see class notes)

QUESTION 19. Find all positive prime integers, say p, such that $p \mid (389^p + 1)$. (see class notes)

QUESTION 20. Let $m > 1$ be an integer and $f(n) = n^m + a_{m-1}n^{m-1} + ... + a_1 n + a_0$, where all the $a_i' s$ are integers and $n \in Z$. Given $f(b_1) = f(b_2) = 22$ for some distinct $b_1, b_2 \in Z$. Prove that $f(k) \neq 23$ for every $k \in Z$. (see class notes)

QUESTION 21. Prove that for each integer $n > 1$, $(2^n - 1)$ is never a factor of $x^2 + 1$ for every $x \in Z$. (see class notes)

QUESTION 22. Let $n, m \geq 1$ be positive integers and $x \in Z^+$. Show that $3^n + 3^m + 1 \neq x^2$;(i.e., Show that $3^n + 3^m + 1$ is never a perfect square.)

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