

HW I

Ayman Badawi

QUESTION 1. 1) Let $x \in Z_{15}$. Solve $6x = 9$ over planet Z_{15} (i.e., Find $x \in Z_{12}$ such that $6x \equiv 9\ (mod\ 15)$)
(**HINT: you must have exactly 3 solutions**)

2) Let $x \in Z$. Solve $6x = 9$ over planet Z (i.e., Find $x \in Z$ such that $6x \equiv 9\ (mod\ 15)$)
(**HINT: 4 is the smallest solution to (1) above and** $d(gap\ \ between\ two\ consecutive\ solutions) = 5$**. Hence** $x \in \{4 + 5k \mid k \in Z\}$)

3) Let $x \in Z$ such that $x\ (mod\ 11) = 2$, $x\ (mod\ 7) = 3$, and $x\ (mod\ 10) = 1$. Find all values of x over Planet Z.
(**HINT: by CRT, 101 is the smallest positive solution and** $m = m_1m_2m_3(gap\ \ between\ two\ consecutive\ solutio$ 770**. Hence** $x \in \{101 + 770k \mid k \in Z\}$)

4) Let $x \in Z$ such that (a) $2x\ (mod\ 12) = 4$, (b) $x\ (mod\ 8) = 4$. Find all values of x over planet Z.
(**HINT: CRT is not applicable, see class notes. First we solve for** x **over** Z **using (a):** $x \in \{2+6m \mid m \in Z\}$ **and 6 =** $d_1 = 12/gcd(2, 12)(gap\ \ between\ two\ consecutive\ solutions)$**. Now we substitute** $2+6m$ **for** x **in (b), and we solve for** m**. So we solve for** m **where** $2+6m\ (mod 8) = 4$**. Hence we solve for** m **where** $6m(mod 8) = 2$**.** $m \in \{3 + 4n \mid n \in Z\}$**, where** $4 = d_2 = 8/gcd(6, 8)(gap\ \ between\ two\ consecutive\ solutions)$**. Since** $m = 3$ **is the smallest postive integer and** $x = 2 + 6m$**, we conclude** $x = 2 + 6(3) = 20$**. Now let us think about the gap** d **between two consecutive solutions (two consecutive values of** x**). From (a)** d **must be multiple of** $d_1 = 12/gcd(2, 12) = 6$**. From (b)** d **must be a multiple of** $8/gcd(1, 8) = 8$**. Thus** $d = LCM[6, 8] = 24$**. Thus the solution set over PLANET** Z **is** $\{20 + 24k \mid k \in Z\}$**.**)

5) Let $x \in Z$ such that (a) $6x\ (mod\ 9) = 3$, (b) $x\ (mod\ 15) = 7$. Find all values of x over planet Z.
(**HINT: CRT is not applicable, see class notes. As in (4), you will end up trying to solve for** m **where** $3m\ (mod\ 15) = 5$ **Since gcd(3, 15) is not a factor of** 5**, we have no solutions for** m**. Thus no such x** $\in Z$ **exists.**
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6) Find $\phi(150)$.

7) Let $M = \{1 \le a < 210 \mid gcd(a, 210) = 1\}$. Find $|M|$.
(**HINT:** $|M| = \phi(210)$**. So calculate** $\phi(210)$**.**)

8) (Nice Question) Find all ordered pairs (x, y) over Z_{12} such that $9x + 2y = 5$ in Z_{12} (i.e., $x, y \in Z_{12}$)
(**HINT: I believe that you will end up with 12 ordered pairs (x, y). Process of thinking is nice!**)

8) (Nice Question) Find all ordered pairs (x, y) over Z such that $9x + 2y = 5$ in Z_{12} (i.e., $x, y \in Z$ such that $9x + 2y \equiv 5\ (mod\ 12)$)
(**HINT: The solution** $\{(3 + 4m_1, 1 + 6m_2) \mid m_1, m_2 \in Z\} \cup \{(1 + 4n_1, 2 + 6n_2) \mid n_1, n_2 \in Z\}$)

9) Find all $x \in Z_{21}$ such that $x^3 - x = x^3 + 20x = 0$ in Z_{21}.
(**HINT: You will be able to find Exactly 9 different roots in** Z_{21}**, note that you may view** Z_{21} **as** Z_3XZ_7**, see class notes**)

10) Find all triplets (x, y, x) over Z_6 such that $4x + 2y + 3z = 5$.
(**hint: There are 24 different triplets**, see class notes)

11) Find all x values over Z where $x\ (mod\ 6) = 2$, $x\ (mod\ 12) = 2$, $x\ (mod\ 5) = 3$, and $x\ (mod\ 7) = 2$.
(**hint: Note that** $12 \mid (x-2)$ **implies that** $6 \mid (x-2)$**. Thus we only need to find** x **where** $x\ (mod\ 12) = 2$, $x\ (mod\ 5) = 3$**, and** $x\ (mod\ 7) = 2$ **. Clearly the CRT is applicable, find the smallest positive integer solution** x_0**, where** $1 \le x_0 < 420$**. Hence the solution set over** Z **is** $\{x_0 + 420k \mid k \in Z\}$**.**)

12) Let n be an odd positive integer. Prove that $\phi(2n) = \phi(n)$.
(**hint: Note that** $gcd(2,n) = 1$**. Hence by class notes,** $\phi(2n) = \phi(2)\phi(n) = 1 \cdot \phi(n) = \phi(n)$**.**)

13) Find $5^{4803}\ (mod\ \ 42)$ (I guess the answer is 41)

14) Let p be a prime positive integer. Prove that $\sum_{d|p^n} \phi(d) = p^n$ for every $n \ge 1$.
(**hint: Abstract algebra is not needed here. Note** $\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1)$**. Use Math induction on** n**. Let** $n = 1$**. T Hence** $\sum_{d|p} \phi(d) = \phi(1) + \phi(p) = 1 + p - 1 = p$**. Assume** $\sum_{d|p^n} \phi(d) = p^n$ **for some** $n \ge 1$**. We prove that** $\sum_{d|p^{(n+1)}} \phi(d) = p^{(n+1)}$**. Note that** $\sum_{d|p^{(n+1)}} \phi(d) = \sum_{d|p^n} \phi(d) + \phi(p^{(n+1)})$ **. Since** $\sum_{d|p^n} \phi(d) = p^n$**, we conclude** $\sum_{d|p^{(n+1)}} \phi(d) = \sum_{d|p^n} \phi(d) + \phi(p^{(n+1)}) = p^n + p^{(n+1)} - p^n = p^{(n+1)}$)

15) Let $n >= 2$ be a positive integer. Prove there are n consecutive numbers such that none of them is prime.
(**hint: Let** $N_2 = (n+1)! + 2$**, is not prime since** 2 **is a factor of** N_2**;** $N_3 = (n+1)! + 3$**, is not prime since** 3 **is a factor of** N_3**;** $N_4 = (n+1)! + 4$**, is not prime since** 4 **is a factor of** N_4**;** $N_5 = (n+1)! + 5$**, is not prime since** 5 **is a factor of** N_5**; ...,** $N_{n+1} = (n+1)! + (n+1)$**, is not prime since** $(n+1)$ **is a factor of** N_{n+1}**. Hence** $N_2, .., N_{n+1}$ **are consecutive** n **integers and none of them is prime.**)

16) Find two consecutive integers x, y such that $125 = x^2 - y^2$. Can you find two integers w, v such that $w - v = 5$ and $125 = w^2 - v^2$.
(**Hint: Let** $x = (125+1)/2), y = (125-1)/2$**. Let** $w = (25+5)/2, v = (25-5)/2)$)

17) Find two integers x, y such that $64 = x^2 - y^2$. Can you find two integers w, vy such that $w - v = 4$ and $64 = w^2 - v^2$.
(**Hint: Let** $x = 16 + 1 = 17, y = 16 - 1 = 15$**. Let** $w = (16+4)/2 = 10, v = (16-4)/2 = 6)$

18) Make sure you know the proofs in your class notes (just the one LABELED: Exams)

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: `abadawi@aus.edu, www.ayman-badawi.com`