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On weakly semiprime ideals of commutative rings

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Abstract Let R be a commutative ring with identity $1 \neq 0$ and let I be a proper ideal of R. D. D. Anderson and E. Smith called I *weakly prime* if $a, b \in R$ and $0 \neq ab \in I$ implies $a \in I$ or $b \in I$. In this paper, we define I to be *weakly semiprime* if $a \in R$ and $0 \neq a^2 \in I$ implies $a \in I$. For example, every proper ideal of a quasilocal ring (R, M) with $M^2 = 0$ is weakly semiprime. We give examples of weakly semiprime ideals that are neither semiprime nor weakly prime. We show that a weakly semiprime ideal of R that is not semiprime is a nil ideal of R. We show that if I is a weakly semiprime ideal of R that is not semiprime and 2 is not a zero-divisor of of R, then $I^2 = \{0\}$ (and hence $i^2 = 0$ for every $i \in I$). We give an example of a ring R that admits a weakly semiprime ideal I that is not semiprime where $i^2 \neq 0$ for some $i \in I$. If $R = R_1 \times R_2$ for some rings R_1, R_2, then we characterize all weakly semiprime ideals of R that are not semiprime. We characterize all weakly semiprime ideals of of $\mathbb{Z}_m$ that are not semiprime. We show that every proper ideal of R is weakly semiprime if and only if either R is von Neumann regular or R is quasilocal with maximal ideal $Nil(R)$ such that $w^2 = 0$ for every $w \in Nil(R)$.

Keywords Primary ideal · Prime ideal · Weakly prime ideal · 2-absorbing ideal · n-absorbing ideal · Semiprime · Weakly semiprime ideal

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1 Introduction

Throughout this paper let R be a commutative ring with identity $1 \neq 0$. Recall that a proper ideal I (i.e., an ideal different from R) of R is called *semiprime* if $a \in R$ and $a^2 \in I$ implies $a \in I$. In this paper, we define a proper ideal I of R to be *weakly semiprime* if $a \in R$ and $0 \neq a^2 \in I$ implies $a \in I$. Recall from (Anderson and Smith 2003) that an ideal I of R is said to be *weakly prime* if $a, b \in R$ and $0 \neq ab \in I$ implies $a \in I$ or $b \in I$. Hence every weakly prime ideal of R is weakly semiprime. However, the converse is not true. For example, the ideal $I = \{0, 8\}$ of $\mathbb{Z}_{16}$ is weakly semiprime that is neither semiprime nor weakly prime. Recently, various generalizations of (weakly) prime ideals are studied in (Anderson and Badawi 2011; Anderson and Smith 2003; Badawi 2007; Badawi and Darani 2013).

Let R be a ring. Then $Nil(R)$ denotes the ideal of nilpotent elements of R. An ideal I of R is said to be a *proper ideal of* R if $I \neq R$. As usual, $\mathbb{Z}$, and $\mathbb{Z}_n$ will denote integers, and integers modulo n, respectively. Some of our examples use the $R(+)M$ construction as in (Huckaba 1988). Let R be a ring and M an R-module. Then $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. Note that $(0(+)M)^2 = 0$; so $0(+)M \subseteq Nil(R(+)M)$.

Among many results in this paper, we show that if I is a weakly semiprime ideal of R that is not semiprime, then $I \subseteq Nil(R)$ (Theorem 2.4). It is shown that if I, J are weakly semiprime ideals of R that are not semiprime and 2 is not a zero-divisor of R, then $I^2 = IJ = \{0\}$ (Theorem 2.8). It is shown that if I is a weakly semiprime ideal of R that is not semiprime and 2 is not a zero-divisor of R, then every ideal $J \subseteq I$ of R is weakly semiprime (and hence $Nil(R)I$ is weakly semiprime) (Theorem 2.11). It is shown that if I is a weakly semiprime ideal of R that is not semiprime and $i^2 \neq 0$ for some $i \in I$, then $2i^2 = i^3 = 0$ and there is an ideal H of R where $\{0\} \neq H^2 \subseteq I$ but $H \nsubseteq I$ (Theorem 2.12). We give an example of a ring R that admits a weakly semiprime ideal I that is not semiprime where $i^2 \neq 0$ for some $i \in I$ (and hence $I^2 \neq \{0\}$) (Example 2.13). If $R = R_1 \times R_2$ where R_1, R_2 are commutative rings with 1, then a complete description of all weakly semiprime ideals of R that are not semiprime is given in Theorems 2.15 and 2.16. If $R = \mathbb{Z}_{p^n}$ where p is a positive prime number and $n \geq 1$ is a positive integer, then it is shown that R admits a weakly semiprime ideal that is not semiprime if and only if $n \geq 4$ is an even integer (Theorem 2.21). It is shown that every proper ideal of R is weakly semiprime if and only if either R is von Neumman regular or R is quasilocal with maximal ideal $Nil(R)$ where $w^2 = 0$ for every $w \in Nil(R)$ (Theorem 2.18).

2 Properties of weakly semiprime ideals

It is clear that every weakly prime ideal of a ring R is semiprime. The following is an example of an infinite ideal I of a commutative ring R such that I is weakly semiprime but I is neither semiprime nor weakly prime.

Example 2.1 Let $M = \{0, 8\}$ and X be an indeterminate. Then $M[X]$ is an ideal of $\mathbb{Z}_{16}[X]$. Let $R = \mathbb{Z}_{16}(+)M[X]$ and let $I = \{0, 8\}(+)M[X]$. Observe that If $y \in R$

and $y^2 \in I$, then $y^2 = (0, 0)$. Hence I is weakly semiprime by definition. Since $(4, 0)^2 = (0, 0) \in I$ and $(4, 0) \notin I$, I is not semiprime. Since $(2, 0)(4, 0) = (8, 0) \in I$ and neither $(2, 0) \in I$ nor $(4, 0) \in I$, I is not weakly prime.

One can easily verify that the ideal $M[X]$ of $\mathbb{Z}_{16}[X]$ is weakly semiprime but it is neither semiprime nor weakly prime.

Definition 2.2 Let I be a weakly semiprime ideal of a ring R and $a \in R$. We say a is an *unbreakable-zero element* of I if $a^2 = 0$ and $a \notin I$.

Theorem 2.3 *Let I be a weakly semiprime ideal of a ring R and suppose that a is an unbreakable-zero element of I. Then $(a + i)^2 = (a - i)^2 = 0$.*

Proof Let $i \in I$. Since $(a+i)^2 = a^2+2ai+i^2 = 0+2ai+i^2 \in I$ and $a \notin I$, we have $a+i \notin I$. Thus $(a+i)^2 = 0$. Similarly, since $(a-i)^2 = a^2-2ai+i^2 = 0-2ai+i^2 \in I$ and $a \notin I$, we have $a - i \notin I$. Thus $(a - i)^2 = 0$. □

Theorem 2.4 *Let I be a weakly semiprime ideal of a ring R that is not semiprime. Then $I \subseteq Nil(R)$.*

Proof Since I is weakly semiprime that is not semiprime, we conclude that I has an unbreakable-zero element, say a. Let $i \in I$. Then $(a+i)^2 = 0$ by Theorem 2.3. Since $a \in Nil(R)$ and $(a + i) \in Nil(R)$, we have $i \in Nil(R)$. Thus $I \subseteq Nil(R)$. □

Theorem 2.5 *Let I be a weakly semiprime ideal of a ring R that is not semiprime. If $char(R) = 2$ (i.e., $1 + 1 = 0 \in R$) or 2 is not a zero-divisor of R, then $i^2 = 0$ for every $i \in I$.*

Proof Since I is weakly semiprime that is not semiprime, we conclude that I has an unbreakable-zero element, say a. Let $i \in I$. Suppose that $char(R) = 2$. Since $(a + i)^2 = 0$ by Theorem 2.3, we have $(a + i)^2 = a^2 + i^2 = 0 + i^2 = 0$. Suppose that $char(R) \neq 2$ and 2 is not a zero-divisor of R. Then $(a + i)^2 + (a - i)^2 = 0$ by Theorem 2.3. Hence $(a + i)^2 + (a - i)^2 = 2i^2 = 0$. Since 2 is not a zero-divisor of R, we conclude that $i^2 = 0$. □

Theorem 2.6 *Let J be a proper ideal of R and suppose that $char(R) = 2$ or 2 is not a zero-divisor of R. The following statements are equivalent:*

(1) I is weakly semiprime that is not semiprime.
(2) If $x^2 \in I$ for some $x \in R$, then $x^2 = 0$.

Proof (1) $\Rightarrow$ (2). Let $x \in R$ and suppose that $x^2 \in I$. Then either $x^2 = 0$ or $x \in I$. If $x \in I$, then $x^2 = 0$ by Theorem 2.5. (2) $\Rightarrow$ (1). It is clear by the definition of weakly semiprime. □

Theorem 2.7 *Let I be a weakly semiprime ideal of a ring R that is not semiprime and suppose that 2 is not a zero-divisor of R. If b is an unbreakable-zero element of I, then $bI = \{0\}$.*

Proof Let $i \in I$. Since $(b + i)^2 = 0$ by Theorem 2.3 and $i^2 = 0$ by Theorem 2.7, we have $(b + i)^2 = b^2 + 2bi + i^2 = 0 + 2bi + 0 = 0$. Hence $2bi = 0$. Since 2 is not a zero-divisor of R, we conclude that $bi = 0$. □

Theorem 2.8 *Let J, I be weakly semiprime ideals of a ring R that are not semiprime and suppose that 2 is not a zero-divisor of R. Then $J^2 = I^2 = IJ = \{0\}$.*

Proof Let $a, b \in I$. Since $a + b \in I$ and 2 is not a zero-divisor of R, $(a + b)^2 = a^2 + 2ab + b^2 = 0$ by Theorem 2.5. Since $a^2 = b^2 = 0$ by Theorem 2.5, we have $2ab = 0$. Since 2 is not a zero-divisor element of R, $ab = 0$. Thus $I^2 = \{0\}$. Similarly, $J^2 = \{0\}$. Now, let $a \in J$ and $b \in I$. Then $a^2 = b^2 = 0$ by Theorem 2.5. Suppose that $a \in I$. Since $I^2 = \{0\}$ and $a, b \in I$, we have $ab = 0$. Suppose that $a \notin I$. Then a is an unbreakable-zero element of I. Hence $ab = 0$ by Theorem 2.7. Thus $JI = \{0\}$. □

The following is an example of an ideal I of a ring R where $I^2 = \{0\}$ but I is not weakly semiprime.

Example 2.9 Let $I = \{0, 4, 8, 12\} \subset R = \mathbb{Z}_{16}$. Then I is an ideal of R and $I^2 = \{0\}$. Since $2^2 \in I$ and $2 \notin I$, I is not weakly semiprime.

Theorem 2.10 *Let I be a weakly semiprime ideal of a ring R that is not semiprime and suppose that 2 is not a zero-divisor of R. Let J be an ideal of R. Then $J^2 \subseteq I$ if and only if $J^2 = \{0\}$.*

Proof Let $a, b \in J$. Since $a^2, b^2, (a+b)^2 \in I$. We have $a^2 = b^2 = (a+b)^2 = 0$ by Theorem 2.6. Thus $(a + b)^2 = 2ab = 0$. Since 2 is not a zero-divisor element of R, we have $ab = 0$. Thus $J^2 = \{0\}$. □

Theorem 2.11 *Let I be a weakly semiprime ideal of a ring R that is not semiprime and suppose that $char(R) = 2$ or 2 is not a zero-divisor of R. Let J be an ideal of R such that $J \subseteq I$. Then J is a weakly semiprime ideal of R. Thus if K is an ideal of R, then KI is a weakly semiprime ideal of R. In particular, $Nil(R)I$ is a weakly semiprime ideal of R.*

Proof Let $x \in R$ and suppose that $x^2 \in J$. Then $x^2 \in I$. Hence $x^2 = 0$ by Theorem 2.6. □

Example 2.13 shows that the hypothesis "2 is not a zero-divisor element" in the previous Theorems is crucial. But first we have the following result.

Theorem 2.12 *Let I be a weakly semiprime ideal of a ring R that is not semiprime and suppose that $i^2 \neq 0$ for some $i \in I$. Then:*

(1) 2 is a nonzero zero-divisor of R, $2i^2 = 0$ and $i^3 = 0$.
(2) If a is an unbreakable-zero element of I, then $2a \in I$, $2ai \neq 0$ (and hence $aI \neq \{0\}$), and $4ai = 0$.
(3) There is an ideal H of R where $\{0\} \neq H^2 \subseteq I$ but $H \nsubseteq I$.

Proof (1) Since $i^2 \neq 0$, 2 is a nonzero zero-divisor of R by Theorem 2.5. Since I is weakly semiprime that is not semiprime, we conclude that I has an unbreakable-zero element, say b. Hence $(b + i)^2 + (b - i)^2 = 2i^2 = 0$ by Theorem 2.3. Since $(b+i)^2 = 2bi + i^2 = 0$ and $2i^2 = 0$, we have $i(b+i)^2 = 2bi^2 + i^3 = 0 + i^3 = 0$.

(2) Let a be an unbreakable-zero element of I. Since $(a+i)^2 = 2ai + i^2 = 0$ and $2i^2 = 0$, we have $2(a+i)^2 = 4ai + 2i^2 = 4ai = 0$. Since $(a+i)^2 = 2ai + i^2 = 0$ and $i^2 \neq 0$, we have $2ai \neq 0$. Since $4ai = 0$ and $i^2 \neq 0$, we have $0 \neq (2a+i)^2 = i^2 \in I$. Thus $(2a+i) \in I$. Since $i \in I$ and $(2a+i) \in I$, we have $2a \in I$.
(3) Let a be an unbreakable-zero element of I and consider the ideal $H = (a, 2i)$ of R. Since $a^2 = 0$ and $2i^2 = 0$, $H^2 = (a, 2i)^2 = (2ai) \subset I$. Since $2ai \neq 0$ by (1), $H^2 = (2ai)$ is a nonzero ideal of R that is contained in I. Since $a \notin I$, $H \nsubseteq I$.

□

In the following example, we show that the hypothesis "2 is not a zero-divisor element" in the previous Theorems is crucial.

Example 2.13 Let $A = \mathbb{Z}_{16}[X]$. Then $J = (X^2 + 8X)$ and $L = (X, 8)$ are ideals of A and $J \subset L$. Let $R = A/J$, $I = L/J$, and $x = X + J \in I$. Then:

(1) $Nil(R) = (2, X)/J$.
(2) $2x^2 = 0$, $x^2 \neq 0$, and $x^3 = 0$.
(3) I is a weakly semiprime ideal of R that is nether semiprime nor weakly prime.
(4) If $b \in R$ is an unbreakable-zero element of I, then $2b \in I$, $2bx \neq 0$ (and hence $bI \neq \{0\}$), and $4bx = 0$.
(5) $H = (J + (4, 2x))/J$ is an ideal of R where $\{0\} \neq H^2 = (J + (8X))/J \subset I$ but $H \nsubseteq I$ (see Theorem 2.12).
(6) $I^2 \neq \{0\}$ and $I^2 \subseteq I$ is not a weakly semiprime ideal of R (compare it with Theorem 2.8 and Theorem 2.11).
(7) $Nil(R)I$ is not a weakly semiprime ideal of R (compare it with Theorem 2.11).

Proof (1) It is clear by construction of R.
(2) By construction of I, $X^2 \notin J$. Thus $x^2 \neq 0$ in R. It is clear that $2(8X + X^2) = 2X^2 \in J$. Thus $2x^2 = 0$ in R. Since $2X^2 \in J$, $bX^2 \in J$ for every $b \in Nil(\mathbb{Z}_{16})$. Since $8X^2 \in J$ and $X(8X + X^2) = 8X^2 + X^3 \in J$, we have $X^3 \in J$. Hence $x^3 = 0$ in R.
(3) Observe that $I \subset Nil(R)$ by construction. Suppose that $m^2 \in I$ and $m^2 \neq 0$. Hence $m = (aX + 2b) + J \in Nil(R)$ for some $a, b \in \mathbb{Z}_{16}$. Since $\mathbb{Z}_{16} \cap L = \{0, 8\}$ and $d^2 = 8$ has no solutions in $\mathbb{Z}_{16}$, we conclude that $m^2 = [(aX + 2b) + J]^2 \in I$ if and only if $(aX + 2b)^2 \in L$ if and only if $(2b)^2 = 0$ in $\mathbb{Z}_{16}$. Thus $2b \in \{0, 4, 8, 12\}$. Suppose that $a \in Nil(\mathbb{Z}_{16})$. If $a = 0$, then $(aX + 2b)^2 = (2b)^2 = 0$. If $a \neq 0$, then it is easily verified that $(aX + 2b)^2 = a^2X^2 \in J$. Hence if $a \in Nil(\mathbb{Z}_{16})$, then $m^2 = 0$. Thus if $m^2 \neq 0$, then $a \notin Nil(\mathbb{Z}_{16})$. Thus suppose that $a \notin Nil(\mathbb{Z}_{16})$. If $2b \in \{4, 12\}$, then $(aX + 2b)^2 = a^2X^2 + 8X$ in A. Since the element $(X^2 + 8X) \in J$, $a^2X^2 + 8X = a^2(X^2 + 8X) \in J$ (note that $a^2 8 = 8$ in $\mathbb{Z}_{16}$). Thus if $2b \in \{4, 12\}$, then $m^2 = 0$ in R. Hence suppose that $2b \in \{0, 8\}$. Then $(aX + 2b)^2 = a^2X^2$ in A. Since $X^2 \notin J$ and a^2 is a unit of $\mathbb{Z}_{16}$, $a^2X^2 \notin J$. Thus $0 \neq m^2 \in I$ if and only if a is a unit of $\mathbb{Z}_{16}$ and $2b \in \{0, 8\}$. Since $aX, aX + 8 \in L$ for every unit a of $\mathbb{Z}_{16}$, we conclude that $0 \neq m^2 \in I$ implies $m \in I$. Thus I is a weakly semiprime ideal of R. Since $(4 + J)^2 = 0$ in R and $4 + J \notin I$, I is not semiprime. Since $0 \neq (4 + J)(2 + J) = 8 + J \in I$ but neither $(4 + J) \in I$ nor $(2 + J) \in I$, I is not weakly prime.

(4) It is clear by Theorem 2.12(2).
(5) Since $x^2 \neq 0$ in R and $4 + J$ is an unbreakable-zero element of I, the claim is clear by Theorem 2.12(3).
(6) It is clear that $I^2 = (8X, X^2)/J \neq \{0\}$. Since $x^2 \in I^2 = (8X, X^2)/J$ and $x \notin I^2$, I^2 is not weakly semiprime.
(7) $K = Nil(R)I = (2X, X^2)/J$. Since $0 \neq x^2 \in K$ and $x \notin K$, K is not weakly semiprime. □

Let I be a weakly semiprime ideal of a ring R_1 and J be a weakly semiprime ideal of R_2. Then $I \times J$ needs not be a weakly semiprime ideal of $R_1 \times R_2$ as we will show in the following example.

Example 2.14 Let R and I be as in Example 2.13 and let $A = R \times \mathbb{Z}_{16}$. Then $J = \{0, 8\}$ is a weakly semiprime ideal of $\mathbb{Z}_{16}$. We show that $I \times J$ is not a weakly semiprime ideal of A. For $0 \neq (X + J, 4)^2 = (X^2 + J, 0) \in I \times J$ but $(X + J, 4) \notin I \times J$.

Theorem 2.15 *Let $R = R_1 \times R_2$ where R_1, R_2 are commutative rings with identity and let J be a proper ideal of R. The following statements are equivalent:*

(1) J is a weakly semiprime ideal of R that is not semiprime such that $x^2 = (0, 0)$ for every $x \in J$.
(2) $J = I_1 \times I_2$ where I_1, I_2 are weakly semiprime ideals of R_1, R_2 respectively and I_1 is not semiprime or I_2 is not semiprime and $a^2 = b^2 = 0$ for every $a \in I_1$ and for every $b \in I_2$.

Proof (1) ⇒ (2). We know that $J = I_1 \times I_2$ for some ideals I_1, I_2 of R_1, R_2 respectively. Suppose that $0 \neq a^2 \in I_1$ for some $a \in R_1$ and $0 \neq b^2 \in I_2$ for some $b \in R_2$. Then $(0, 0) \neq (a^2, b^2) \in J$. Since J is weakly semiprime, we have $(a, b) \in J$. Thus $a \in I_1$ and $b \in I_2$. Thus I_1 is a weakly semiprime ideal of R_1 and I_2 is a weakly semiprime ideal of R_2. Since $x^2 = (0, 0)$ for every $x \in J$, we have $a^2 = b^2 = 0$ for every $a \in I_1$ and for every $b \in I_2$. Since J is not semiprime, J has an unbreakable-zero element, say $(c, d) \in R$. Hence c is an unbreakable-zero element of I_1 or d is an unbreakable-zero element of I_2. Thus I_1 is not semiprime or I_2 is not semiprime.
(2) ⇒ (1). It can be easily verified and it is left to the reader. □

Theorem 2.16 *Let $R = R_1 \times R_2$ where R_1, R_2 are commutative rings and let J be a proper ideal of R. The following statements are equivalent:*

(1) J is a weakly semiprime ideal of R that is not semiprime such that $x^2 \neq (0, 0)$ for some $x \in J$.
(2) $J = I_1 \times I_2$ where (I_1 is a weakly semiprime ideal of R_1 that is not semiprime such that $a^2 \neq 0$ for some $a \in I_1$ and I_2 is a semiprime ideal of R_2 such that $b^2 = 0$ for every $b \in I_2$) or (I_2 is a weakly semiprime ideal of R_2 that is not semiprime such that $b^2 \neq 0$ for some $b \in I_2$ and I_1 is a semiprime ideal of R_1 such that $a^2 = 0$ for every $a \in I_1$).

Proof (1) ⇒ (2). We know that $J = I_1 \times I_2$ for some ideals I_1, I_2 of R_1, R_2 respectively. Then I_1, I_2 are weakly semiprime ideals of R_1, R_2 respectively by the first part proof of Theorem 2.15. Since J is a weakly semiprime ideal of R that is not semiprime, we conclude that I_1 a weakly semiprime ideal of R_1 that is not semiprime or I_2 is a weakly semiprime ideal of R_2 that is not semiprime. We consider two cases. *Case one* Suppose that I_1 a weakly semiprime ideal of R_1 that is not semiprime. We show that I_2 is a semiprime ideal of R_2 such that $b^2 = 0$ for every $b \in I_2$. Hence I_1 has unbreakable-zero element $c \in R_1$. Let $b \in I_2$. Since $(c, b)^2 = (c^2, b^2) = (0, b^2) \in J$ and $c \notin I_1$, we conclude that $b^2 = 0$. Since $x^2 \neq (0, 0)$ for some $x \in J$ and $b^2 = 0$ for every $b \in I_2$, we conclude that there is an $h \in I_1$ such that $h^2 \neq 0$. Let $f \in R_2$ and suppose that $f^2 \in I_2$. Since $(0, 0) \neq (h, f)^2 = (h^2, f^2) \in J$ and J is a weakly semiprime of R, we conclude that $f \in I_2$. Thus I_2 is a semiprime ideal of R_2 such that $b^2 = 0$ for every $b \in I_2$. *Case two* Suppose that I_2 a weakly semiprime ideal of R_2 that is not semiprime. Then by similar argument as in case one, we conclude that I_1 is a semiprime ideal of R_1 such that $a^2 = 0$ for every $a \in I_1$. (2) $\Rightarrow$ (1). It can be easily verified and it is left to the reader. □

Theorem 2.17 *Every nil ideal of R is weakly semiprime if and only if $w^2 = 0$ for every $w \in Nil(R)$.*

Proof Suppose that every nil ideal of R is weakly semiprime. Let $w \in Nil(R)$ and suppose that $w^2 \neq 0$ for some $w \in Nil(R)$. Since $J = w^2R$ is weakly semiprime and $0 \neq w^2 \in J$, $w \in J$. Hence $w^2a = w$ for some $a \in R$. Thus $w(wa - 1) = 0$. Since $aw - 1$ is a unit of R, $w = 0$, a contradiction. Thus $w^2 = 0$ for every $w \in Nil(R)$.

Conversely, suppose that $w^2 = 0$ for every $w \in Nil(R)$. Let I be a nil ideal of R. Then I is weakly semiprime by Theorem 2.6. □

Recall that an element $x \in R$ is said to be *von Neumann regular* if $ux^2 = x$ for some $u \in R$. If each element of R is von Neumann regular, then R is called *von Neumann regular* . For a recent article on von Neumann regular elements of a ring R see (Anderson and Badawi 2012). If R has exactly one maximal ideal, then we say that R is *quasilocal*. A ring R is said to be *reduced* if $Nil(R) = \{0\}$. It is known that if R is von Neumann regular, then R is reduced. We have the following result.

Theorem 2.18 *The following statements are equivalent:*

(1) Every proper ideal of R is weakly semiprime.
(2) Either R is von Neumann regular (and hence R is reduced) or R is quasilocal with maximal ideal $Nil(R)$ such that $w^2 = 0$ for every $w \in Nil(R)$.

Proof (1) $\Rightarrow$ (2). Since every nil ideal of R is weakly semiprime, we have $w^2 = 0$ for every $w \in Nil(R)$ by Theorem 2.17. Hence let $x \in R \backslash Nil(R)$. If x is a unit of R, then x is von Neumann regular. Hence assume that x is not a unit of R. Since $I = x^2R$ is weakly semiprime and $0 \neq x^2 \in I$, $x \in I$. Thus $x = ux^2$ for some $u \in R$. Hence x is a von Neumann regular element of R. Since each element y of R is either nilpotent with $y^2 = 0$ or y is von Neumann regular, we conclude that either R is von Neumann regular or R is quasilocal with maximal ideal $Nil(R)$ such that $w^2 = 0$ for every $w \in Nil(R)$ by [Anderson and Badawi (2012), Theorem 2.4(1)]. (2) $\Rightarrow$ (1). It is clear by the definition of von Neumann regular and by Theorem 2.17. □

In view of Theorem 2.18, we have the following result.

Corollary 2.19 *Let R be a reduced ring. The following statements are equivalent:*

(1) Every proper ideal of R is weakly semiprime.
(2) Every proper ideal of R is semiprime.
(3) R is von Neumann regular.

Recall that and ideal I of R is said to be *nontrivial* if $I \neq \{0\}$.

Theorem 2.20 *Assume that either $n = 2$ or $n \geq 3$ is an odd integer and let $R = \mathbb{Z}_{p^n}$ where p is a positive prime integer. Then a nontrivial proper ideal I of R is weakly prime if and only if I is prime.*

Proof Assume $n = 2$. Since pR is the only nontrivial ideal of R. The claim is clear. Hence assume that $n \geq 3$ is an odd integer. Suppose that I is a nontrivial weakly prime ideal of R. Then $I = p^k R$ for some integer k, $1 \leq k \leq n-1$. Suppose that k is even. Then $0 \neq (p^{k/2})^2 \in I$ but $p^{k/2} \notin I$, a contradiction. Thus assume that k is odd. Since n is odd, we have $1 \leq k \leq n-2$. Since $0 \neq (p^{(k+1)/2})^2 \in I$, we have $p^{(k+1)/2} \in I$. But $p^{(k+1)/2} \in I$ if and only if $k = 1$. Hence $I = pR$ is a prime ideal of R. The converse is clear. □

Theorem 2.21 *Let $n \geq 2$ be a positive integer, p be a positive prime integer, and let $R = \mathbb{Z}_{p^n}$. The following statements are equivalent:*

(1) $n \geq 4$ is an even integer.
(2) R has a nontrivial weakly semiprime ideal that is not semiprime.
(3) R has a unique nontrivial weakly semiprime ideal that is not semiprime.
(4) $I = p^{n-1}R$ is the only nontrivial weakly semiprime ideal of R that is not semiprime.

Proof (1) $\Rightarrow$ (2). Let $I = p^{n-1}R$. Then let $x \in R$. Since $n \geq 4$ is an even integer, $x^2 \in I$ if and only if $x^2 = 0$ in R. Thus I is weakly semiprime by definition. Since $n \geq 4$ is an even integer, $(p^{n/2})^2 = 0$ in R but $p^{n/2} \notin I$. Hence I is not semiprime. (2) $\Rightarrow$ (3). Since R has a nontrivial weakly semiprime ideal that is not semiprime, we conclude that $n \geq 4$ is an even integer by Theorem 2.20. Let $1 \leq k < n$. Since $(R, +)$ is a cyclic group under addition, there is exactly one ideal of order p^k, namely $p^{n-k}R$. Thus suppose that $I = p^{n-k}R$ is weakly semiprime that is not semiprime for some $2 \leq k < n$. Then either $n - k$ is an even integer or $n - k$ is an odd integer. If $n - k$ is an even integer, then $0 \neq (p^{(n-k)/2})^2 \in I$ but $p^{(n-k)/2} \notin I$, a contradiction. If $m = n - k$ is an odd integer, then $0 \neq (p^{(m+1)/2})^2 \in I$ but $p^{(m+1)/2} \notin I$ (note that $(m + 1)/2 < n - k$), a contradiction again. Hence $I = p^{n-1}R$ is the only nontrivial weakly semiprime ideal of R that is not semiprime. (3) $\Rightarrow$ (4). Since R has a nontrivial weakly semiprime ideal that is not semiprime, we conclude that $n \geq 4$ by Theorem 2.20. It is shown earlier in the proof that $I = p^{n-1}R$ is the only nontrivial weakly semiprime ideal of R that is not semiprime. (4) $\Rightarrow$ (1). It is clear by Theorem 2.20. □

Corollary 2.22 *Let $n \geq 2$ be a positive integer, p be a positive prime integer, $R = \mathbb{Z}_{p^n}$, and let I be a nontrivial proper ideal of R. The following statements are equivalent:*

(1) $n = 2$ or $n \geq 3$ is an odd integer.
(2) I is weakly semiprime if and only if I is prime.

Proof In view of Theorem 2.20 and Theorem 2.21, the claim is clear. □

Remark 2.23 Assume that $R = R_1 \times \cdots \times R_k$, where $R_1, \ldots, R_k$ are commutative rings with 1 and $k \geq 2$. It should be clear that if $I = I_1 \times \cdots \times I_k$ is a weakly semiprime ideal of R that is not semiprime where each I_i is an ideal of R_i, then $I_i \neq R_i$ for each i, $1 \leq i \leq k$.

Theorem 2.24 *Let $m = p_1^{n_1}, \ldots, p_k^{n_k}$ where the p_i's are distinct positive prime integers, the n_i's ≥ 1 are positive integers, and $k \geq 2$. Let $R = \mathbb{Z}_m = R_1 \times \cdots \times R_k$ where each $R_i = \mathbb{Z}_{p_i^{n_i}}$. Then R admits a nontrivial weakly prime ideal that is not semiprime if and only if one of the following two conditions holds:*

(1) There is an i, $1 \leq i \leq k$ such that $n_i \geq 4$ is an even integer.
(2) There are distinct i, j, $1 \leq i, j \leq k$ such that $n_i = 2$ and $n_j \geq 2$.

Proof Suppose that R admits a nontrivial weakly prime ideal, say I, that is not semiprime. Hence $I = I_1 \times \cdots \times I_k$ where each I_i is a weakly semiprime ideal of R_i. Assume that the n_i's are all odd integers. Then each nontrivial I_i is a prime ideal of R_i by Theorem 2.20 and Remark 2.23. Assume that $n_i = 1$ for every i, $1 \leq i \leq k$. Then R is von Neumann regular, and hence every nontrivial weakly semiprime ideal of R is semiprime, a contradiction. Since I is not semiprime, one of the I_i's, say I_1, is weakly semiprime that is not semiprime. Since n_1 is odd, $I_1 = \{0\}$ and $n_1 \geq 3$. Since I is nontrivial, one of the I_i's, say I_2, is nontrivial prime ideal. Hence $I_2 = p_2R_2$ and $n_2 \geq 3$. Now $(0, 0, \ldots, 0) \neq (p_1^{(n_1+1)/2}, p_2, 0, \ldots, 0)^2 \in I$ but $(p_1^{(n_1+1)/2}, p_2, 0, \ldots, 0) \notin I$, a contradiction. Now assume exactly one of the n_i's, say $n_1 = 2$, and each $n_i = 1$, $2 \leq i \leq k$. Then it is easily verified that every weakly semiprime of R is semiprime, a contradiction. Thus one of the two given conditions must hold.

Conversely, assume that one of the n_i's, say $n_1 \geq 4$ is an even integer. Then $I_1 = p_1^{n_1-1}R_1$ is a weakly semiprime ideal of R_1 that is not semiprime by Theorem 2.21. Hence $I = I_1 \times \{0\} \times \cdots \times \{0\}$ is a nontrivial weakly semiprime ideal of R that is not semiprime. Assume that $n_1 = 2$ and $n_2 \geq 2$. Then $I = p_1R_1 \times \{0\} \times \cdots \times \{0\}$ is a nontrivial weakly semiprime ideal of R that is not semiprime. □

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