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Abstract Algebra I

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Task 1: -

Let $(D,*)$ be a non-abelian group with 75 elements. For each prime factor p of |D|, find n_p. Is D simple? Explain?

Solution:

If we rewrite the order of D as:

$|D| = 75 = 5^2 \cdot 3$

Then:

$p = 5, k = 2, m = 3$

Let n_5 be the number of all distinct Sylow-5-subgroups of D.

then:

$\mathrm{P}|(n_p - 1)$ and $n_p|m$

$5|n_5 - 1$ and $n_5|3$

Thus, $n_5 = 1$.

Such subgroup is unique and hence it is normal in D and Therefore D is not simple.

The reason is: $(D,*)$ is called Simple if the trivial groups$\{\{e\},\{D\}\}$ are the only normal subgroup of D.

Again if we rewrite the order of D as:
$|D| = 75 = 3 \cdot 5^2$
Then:
$p = 3, k = 3, m = 25$
Let n_3 be the number of all distinct Sylow-3-subgroups of D.
then:

P$|(n_p - 1)$ and $n_p|m$

$3|(n_3 - 1)$ and $n_3|25$

In this case we have Two scenarios:
First scenario: -
$n_3 = 1.$
Thus, $n_3 = 1.$
Such subgroup is unique and hence it is normal in D and there for D is not simple.

Second scenario: -

$n_3 = 25.$

This we have 25 distinct Sylow-3-Subgroups of D and each one of them is order 3.

Say $\{H_1, H_2 \dots H_{25}\}$

Now lets calculate the order of:

$|H_1 \cup H_2 \cup \dots \dots \dots \dots . \cup H_{25}| = 51$

Thus we must have only on Sylow-5-Subgroup, Such group has 25 elements. And since it is unique it is normal in D.

Thus D is not simple.

Task 2: -

Let D be a group with 40 elements. Assume that D has a normal abelian subgroup with 8 elements.

a) For each prime factor p of |D|, find n_p.

If we rewrite the order of D as:

$|D| = 40 = 5^1 \cdot 2^3$

Then:

$p = 5, k = 1, m = 8$

Let $n_5$be the number of all distinct Sylow-5-subgroups of D.

then:

P$|(n_p - 1)$ and $n_p|m$

$5|(n_5 - 1)$ and $n_5|8$

Thus, $n_5 = 1$.

Thus such subgroup is unique and hence it is normal in D.

The reason is: $(D,*)$ is called Simple if the trivial groups$\{\{e\},\{D\}\}$ are the only normal subgroup of D.

Again if we rewrite the order of D as:
$|D| = 40 = 2^3 \cdot 5$
Then:
$p = 2, k = 3, m = 5$
Let n_2 be the number of all distinct Sylow-2-subgroups of D.
And since it is given that D has normal abelian with 8 elements,
And since n_2 is to order 8 it is unique in D.
Thus $n_2 = 1$.

b) Then prove that D is abelian. Assume that D has a normal abelian subgroup with 8 elements.

To prove that D is abelian we need to show it is cyclic

And we will do that by using the 3rd important result after Lagrange.

Lets cook it☺

From part A: D has one normal subgroups of order 5,

Say k, $K \lhd D$, and $|K| = 5$.

Also given that D has a normal abelian subgroup with 8 elements. Let it be H, $H \lhd D$

Thus $|H| = 8$.

$|H \times K| = \frac{|H||K|}{|H \cap K|}$

And since; $H \cap K = \{e\}$, $|H \cap K| = 1$

Therefore:

$$|H \times K| = \frac{|8| \cdot |5|}{|1|} = 40 = |D|$$

$$D \cong \frac{D}{H} \times \frac{D}{K}$$

$$D \cong K \times H$$

Now, and since K is cyclic with 5 elements it is isomorphic to Z_5.

Since H is normal abelian with 8 elements it is isomorphic to:

Either $H \cong Z_8$

Or $H \cong Z_2 \times Z_4$

Or $H \cong Z_2 \times Z_2 \times Z_2$

Hence, $D \cong Z_8 \times Z_5$ and since $gcd\ (8,5) = 1$

Therefore D is cyclic.

And every cyclic group is abelian.

c) Up to isomorphism classify all abelian groups of order 40.

We rewrite the order of D as:

$|D| = 40 = 2^3 \cdot 5$

We need to find all partitions of 3 and 1

Partitions of 3 are:

$$3 \rightarrow Z_8$$
$$1 + 2 \rightarrow Z_2 \times Z_4$$
$$1 + 1 + 1 \rightarrow Z_2 \times Z_2 \times Z_2$$

and there is only on Partition of 1 which is:

$$1 \rightarrow Z_5$$

Now we will pair up each one of them by direct product and hence the group with 40 elements is Isomorphic to either:

$$D \cong Z_8 \times Z_5 or Z_{40}$$

$$D \cong Z_2 \times Z_4 \times Z_5 \ or Z_2 \times Z_{20}$$

$$D \cong Z_2 \times Z_2 \times Z_2 \times Z_5 \ \ or \ Z_2 \times Z_2 \times Z_{10}$$

THE END

THANK YOU FOR LISTENING