MTH 530

SYLOW THEOREM APPLICATION EXAMPLES

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Outline

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Historical Note

- **Peter Ludvig Mejdell Sylow**, a Norwegian Mathematician that lived between 1832 and 1918.
- He published the Sylow theorems in a brief paper in 1872.
- Sylow stated them in terms of permutation groups.
- Georg Frobenius re-proved the theorems for abstract groups in 1887.
- Sylow applied the theorems to the question of solving algebraic equations and showed that any equation whose Galois group has order a power of a prime p is solvable by radicals.
- Sylow spent most of his professional life as a high school teacher in Halden, Norway, and only appointed to a position at Christiana University in 1898. He devoted 8 years of his life to the project of editing the mathematical works of his countryman Niels Henrik Abel. [1]

Example 1:

A_5 is a simple group of S_5,
How can we find n_p for each prime factor say p of $|A_5|$

Sylow's Theorem

$(D,*)$ is a group, $|D| = p^k m$, where p is prime and $\gcd(p,m) = 1$;

Syl 1	$\forall\, i, 1 \leq i \leq k, \exists$ at least one subgroup of order p^i.
Syl 2	A subgroup of D with p^k elements, we call it a Sylow p-subgroup.
Syl 3	A subgroup of D with p^i elements, $1 \leq i \leq k$, we call it p-subgroup.
Syl 4	If H is a p-subgroup, then H is a subgroup of a Sylow p-subgroup.
Syl 5	Let n_p = # of distinct Sylow p-subgroups. Then $n_p \mid m$ and $p \mid (n_p - 1)$.
Syl 6	A Sylow p-subgroup is normal in D **iff** $n_p = 1$.
Syl 7	$(D,*)$ is called simple group **if** {e} and D are the only normal subgroups of D.

Solution

- First, compute the size of A_5:

$$|A_5| = \frac{5!}{2} = \frac{5*4*3*2*1}{2} = 5*4*3$$

$$= 5*2^2*3 = 60$$

- Let,

$n_5 =$ # of distinct Sylow 5-subgroups
$n_3 =$ # of distinct Sylow 3-subgroups
$n_2 =$ # of distinct Sylow 2-subgroups

Find n_5

- Let, $|A_5| = 5 * 12$

  Using **Syl 5**, we have:

  $n_5|12$ and $5|(n_5 - 1)$

  $\therefore\ n_5 \in \{1, 6\}$

- Since A_5 is a simple group using **Syl 6** and **Syl 7** we have $n_5 \neq 1$ $\rightarrow n_5 = 6$

- Let $K_1, K_2, \ldots, K_6$ be distinct Sylow 5-subgroups, where each consists of 5 elements.

- We have $\bigcap_{i=1}^{6} K_i = (1)$ and the intersection between every pair of Sylow 5-subgroups is (1), since they have a prime order, hence $\left|\bigcup_{i=1}^{6} K_i\right| = (6 * 4) + 1 = 25$ elements

Find n_3

- Let, $|A_5| = 3 * 20$

  Using **Syl 5**, we have:

  $n_3|20$ and $3|(n_3 - 1)$

  $\therefore\ n_3 \in \{1, 4, 10\}$

- Since A_5 is a simple group using **Syl 6** and **Syl 7** we have $n_3 \neq 1$
  $\rightarrow n_3 \in \{4, 10\}$

Find n_3

Remark

Any element of **odd order** in S_5 is an **even permutation**.
Thus all **3-cycle** and **5-cycle** elements are in A_5.

Note

Each distinct Sylow 3-subgroup, consists of 3 elements.
So we need to find the number of **3-cycles** in A_5, and distribute them into distinct Sylow 3-subgroups.

I: 3-cycles in A_5

- $\left.\begin{matrix}(1\ 2\ 3)\\(3\ 2\ 1)\end{matrix}\right\} \to \{(1), (1\ 2\ 3), (3\ 2\ 1)\} = \langle(1\ 2\ 3)\rangle = H_1$
- $\left.\begin{matrix}(1\ 2\ 4)\\(4\ 2\ 1)\end{matrix}\right\} \to \{(1), (1\ 2\ 4), (4\ 2\ 1)\} = \langle(1\ 2\ 4)\rangle = H_2$
- $\left.\begin{matrix}(1\ 2\ 5)\\(5\ 2\ 1)\end{matrix}\right\} \to \{(1), (1\ 2\ 5), (5\ 2\ 1)\} = \langle(1\ 2\ 5)\rangle = H_3$
- $\left.\begin{matrix}(1\ 3\ 4)\\(4\ 3\ 1)\end{matrix}\right\} \to \{(1), (1\ 3\ 4), (4\ 3\ 1)\} = \langle(1\ 3\ 4)\rangle = H_4$
- $\left.\begin{matrix}(1\ 3\ 5)\\(5\ 3\ 1)\end{matrix}\right\} \to \{(1), (1\ 3\ 5), (5\ 3\ 1)\} = \langle(1\ 3\ 5)\rangle = H_5$

II: 3-cycles in A_5

- $\left.\begin{matrix}(1\ 4\ 5)\\(5\ 4\ 1)\end{matrix}\right\} \rightarrow \{(1), (1\ 4\ 5), (5\ 4\ 1)\} = \langle(1\ 4\ 5)\rangle = H_6$
- $\left.\begin{matrix}(2\ 3\ 4)\\(4\ 3\ 2)\end{matrix}\right\} \rightarrow \{(1), (2\ 3\ 4), (4\ 3\ 2)\} = \langle(2\ 3\ 4)\rangle = H_7$
- $\left.\begin{matrix}(2\ 3\ 5)\\(5\ 3\ 2)\end{matrix}\right\} \rightarrow \{(1), (2\ 3\ 5), (5\ 3\ 2)\} = \langle(2\ 3\ 5)\rangle = H_8$
- $\left.\begin{matrix}(2\ 4\ 5)\\(5\ 4\ 2)\end{matrix}\right\} \rightarrow \{(1), (2\ 4\ 5), (5\ 4\ 2)\} = \langle(2\ 4\ 5)\rangle = H_9$
- $\left.\begin{matrix}(3\ 4\ 5)\\(5\ 4\ 3)\end{matrix}\right\} \rightarrow \{(1), (3\ 4\ 5), (5\ 4\ 3)\} = \langle(3\ 4\ 5)\rangle = H_{10}$

III: 3-cycles in A_5

- The number of 3-cycles in A_5 is 20, and these come in inverse pairs, giving us 10 subgroups of size 3

- Since $\bigcap_{i=1}^{10} H_i = (1)$ and the intersection between every pair of Sylow 3-subgroups is (1), since they have a prime order,

  $\rightarrow$ we have 10 distinct subgroups of size 3

  $\rightarrow$ we have 10 Sylow-3 subgroups

  $\therefore\ n_3 = 10$

- In addition,

  $\left|\bigcup_{i=1}^{10} H_i\right| = (10 * 2) + 1 = 21$ elements

Find n_2

- Let, $|A_5| = 2^2 * 15$

  Using **Syl 5**, we have:

  $n_2|15$ and $2|(n_2 - 1)$

  $\therefore\ n_2 \in \{1, 3, 5, 15\}$

- Since A_5 is a simple group using **Syl 6** and **Syl 7** we have $n_2 \neq 1$ $\rightarrow n_2 \in \{3, 5, 15\}$

Find n_2

Remark:

A_5 has **no** 2-cycle or 4-cycle elements, since these are odd permutations.

What are the remaining non-identity elements in A_5?

$$\left|\bigcup_{i=1}^{6} K_i\right| + \left|\bigcup_{j=1}^{10} H_j/(1)\right| = 25 + 20 = 45$$

$$|A_5| - 45 = 60 - 45 = 15$$

- So we have 15 non-identity elements left, of the form $(a\ b)(c\ d)$, such that each has order 2
- Let N_r be distinct Sylow 2-subgroups, where $r = 1 \ldots n_2$ and $\bigcap_{r=1}^{n_2} N_r = (1)$
- We have 3 non-identity elements in every Sylow 2-subgroup

$$\therefore\ n_2 = \frac{15}{3} = 5$$

Subgroups of size 4 in A_5 where each of their non-identity elements has an order of 2

- $\left.\begin{matrix}(1\ 2)(3\ 4)\\(1\ 3)(2\ 4)\\(1\ 4)(2\ 3)\end{matrix}\right\} \rightarrow \{(1),(1\ 2)(3\ 4),(1\ 3)(2\ 4),(1\ 4)(2\ 3)\} = N_1$
- $\left.\begin{matrix}(1\ 2)(4\ 5)\\(1\ 4)(2\ 5)\\(1\ 5)(2\ 4)\end{matrix}\right\} \rightarrow \{(1),(1\ 2)(4\ 5),(1\ 4)(2\ 5),(1\ 5)(2\ 4)\} = N_2$
- $\left.\begin{matrix}(1\ 2)(3\ 5)\\(1\ 3)(2\ 5)\\(1\ 5)(2\ 3)\end{matrix}\right\} \rightarrow \{(1),(1\ 2)(3\ 5),(1\ 3)(2\ 5),(1\ 5)(2\ 3)\} = N_3$
- $\left.\begin{matrix}(1\ 3)(4\ 5)\\(1\ 4)(3\ 5)\\(1\ 5)(3\ 4)\end{matrix}\right\} \rightarrow \{(1),(1\ 3)(4\ 5),(1\ 4)(3\ 5),(1\ 5)(3\ 4)\} = N_4$
- $\left.\begin{matrix}(2\ 3)(4\ 5)\\(2\ 4)(3\ 5)\\(2\ 5)(3\ 4)\end{matrix}\right\} \rightarrow \{(1),(2\ 3)(4\ 5),(2\ 4)(3\ 5),(2\ 5)(3\ 4)\} = N_5$

Example 2:

Let $(D,*)$ be an abelian group with 72 elements, Prove that D has only one subgroup of order 8, say H, and one subgroup of order 9, say K.

Up to isomorphism, classify all abelian groups of order 72.

Solution

- First, let, $|D| = 72 = 8 * 9 = 2^3 * 3^2$
- Using **Syl 1**:
  $\exists$ a Sylow 2-subgroup, H of size 8
  $\exists$ a Sylow 3-subgroup, K of size 9
- Since D is an abelian group, we have:
  $$H \lhd D \text{ and } K \lhd D$$
- Using **Syl 6**, since H and K are both Sylow p-subgroups (where p is prime) and they are both normal, we have:
  $$n_2 = 1 \text{ and } n_3 = 1$$
- Thus we have a unique subgroup $H$of size 8 and a unique subgroup K of size 9

Up to isomorphism classification of all abelian groups of order 72

Partitions of 3	Isomorphism
3	$Z_{2^3} = Z_8$
$1+2$	$Z_2 \times Z_{2^2} = Z_2 \times Z_4$
$1+1+1$	$Z_2 \times Z_2 \times Z_2$

Partitions of 2	Isomorphism
2	$Z_{3^2} = Z_9$
$1+1$	$Z_3 \times Z_3$

Up to isomorphism classification of all abelian groups of order 72

$D \cong Z_8 \times Z_9$
$D \cong Z_8 \times Z_3 \times Z_3$
$D \cong Z_2 \times Z_4 \times Z_9$
$D \cong Z_2 \times Z_4 \times Z_3 \times Z_3$
$D \cong Z_2 \times Z_2 \times Z_2 \times Z_9$
$D \cong Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3$

Conclusion

- The fundamental theorem for finitely generated abelian groups gives us complete information about all finite abelian groups.
- The study of finite non-abelian groups is much more complicated.
- For non-abelian group G, the “converse of Lagrange theorem” does not hold.
- The Sylow theorems give a weak converse. Namely, they show that if d is a power of a prime and $d \mid |G|$, then G does contain a subgroup of order d.
- The Sylow theorems also give some information on the number of such groups and their relationship to each other, which can be very useful in studying finite non-abelian groups.[1]

References

[1] J. B. Fraleigh, *A First Course in Abstract Algebra*, 7th edition, 2003.