

Exam one: MTH 530, Fall 2017

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QUESTION 1. (8 points). Let $(D, *)$ be a group with 85 elements. Use results from Sylow's Theorems to prove that D is cyclic.

Solution: $|D| = 17X5$. $n_{17} \mid 5$ **and** $17 \mid (n_{17} - 1)$. **Hence** $n_{17} = 1$. **Thus** D **has only one subgroup of** D**, say** H**, with 17 elements. Hence** H **is normal in D. Also,** $n_5 \mid 17$ **and** $5 \mid (n_5 - 1)$. **Thus** $n_5 = 1$. **Thus** D **has only one subgroup of** D**, say** K**, with 5 elements. Thus** K **is normal in D. It is clear that** $D = H * K$ **and** $H \cap K = \{e\}$. **Thus** $D \cong H \times K$. **Since** K **and** D **are cyclic and** $gcd(5, 17) = 1$**, we conclude that** D **is cyclic.**

QUESTION 2. (8 points) Let $(D, *)$ be a group with $p^{2017}m$, where p is prime, m is a positive integer, and $gcd(p, m) = 1$. Assume that H is the only subgroup of D with p elements (hence we know that H is a normal subgroup of D). Assume that D/H is cyclic. Prove that D is cyclic.

Solution: $F = D/H =< a * H >$ **for some** $a \in D$. **Let** $k = |a|$ **in D. Since** $|F| = |a * H| = p^{2016}m$**, we conclude** $k = |a| = p^{2016}m$ **OR** $k = |a| = p^{2017}m$. **We show** $k = |a| = p^{2017}m$. **Now,** $< a >$ **is a cyclic subgroup of** D **and** $p \mid k$. **Thus** $< a >$ **has a unique subgroup, say** L**, with** p **elements. Since** H **is the only subgroup of** D **with** p **elements, we conclude that** $H = L$. **Thus** $L = H = \{a^{i_1}, a^{i_2}, ..., a^{i_p} = e\}$. **Let** $d \in D$. **We show that** $d \in< a >$. **Since** F **is cyclic,** $d * H = a^n * H$. **Thus** $d = a^n * h$ **for some** $h \in H$. **Hence** $d = a^n * a^{i_j}$ **for some** $1 \le j \le p$. **Thus** $d = a^{n+i_j} \in< a >$. **Hence** $D =< a >$.

QUESTION 3. (i) **(4 points)**. Is $(Z_2, +) \times (Z_6, +)$ group-isomorphic to $(Z_{12}, +)$? If yes, then prove it. If no, then tell me why not?

Solution: Since $gcd(2, 12) \neq 1$, $Z_2 \times Z_6$ **is not cyclic with 12 elements. However,** Z_{12} **is cyclic with 12 elements. Thus they are not isomorphic.**

(ii) **(4 points)**. Is $(Z_{41}^*, .)$ group-isomorphic to $(U(75), .)$? If yes, then prove it. If no, then tell me why not?

Solution: Since 75 is not of the form $2p^m, p^m$ **for some odd prime p, we know** $U(75)$ **is not cyclic with 40 elements. However,** Z_{41}^* **is cyclic with 40 elements. They are not isomorphic.**

(iii) **(4 points)**. Construct a subgroup of $(Z_4, +) \times (Z_5^*, .)$, say H, such that H has 4 elements, but there is no subgroup N_1 of $(Z_4, +)$ and there is no subgroup N_2 of $(Z_5^*, .)$ such that $H = N_1 \times N_2$.

Solution Let $a = (2, 2)$. **Then** $|a| = 4$. **Now** $L = \{a, a^2, a^3, a^4\} = \{(2, 2), (0, 4), (2, 3), (0, 1)\}$. **If** $L = N_1 \times N_2$**, then by staring at the elements of** L**, we conclude that** $|N_1|$ **is at least** 2 **and** $|N_2| = 4$ **and thus** $|L|$ **is at least** 8**, impossible.**

(iv) **(6 points)**. Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 9 & 8 & 2 & 6 & 5 & 1 \end{pmatrix} \in S_9$. Find $|f|$. Is $f \in A_9$? explain

Solution: No comments!!

QUESTION 4. (6 points). Let $(D, *)$ be a group. Assume that $a * b = b * a$ for some $a, b \in D$, $|a| = n$, and $|b| = m$. Let $u = lcm[n, m]$. Prove that D has a cyclic subgroup with u elements. (Hint: You may need the fact: if $d = gcd(n, m)$, then $gcd(\frac{n}{d}, m) = 1$ OR $gcd(n, \frac{m}{d}) = 1$).

Solution: Assume $gcd(n/d, m) = 1$. **We know** $|a^d| = n/gcd(d, n) = n/d$. **Since** $a^d * b = b * a^d$ **and** $gcd(|b|, |a^d|) = gcd(m, n/d) = 1$**, we know** $|b * a^d| = |b||a^d| = mn/d = LCM[m, n] = u$. **Since** $|b * a^d| = u$**, we conclude that** $< b * a^d >$ **is a cyclic subgroup with** u **elements.**

QUESTION 5. (6 points). Assume $(D, *)$ is a group with p^5 elements for some prime number p. Assume D has a normal cyclic subgroup H with p^4 elements and D has a normal cyclic subgroup F with p elements such that $F \nsubseteq H$. Prove that D is abelian but not cyclic.

Solution: Since $F \nsubseteq H$ **and** F **has** p **elements,** $H \cap F = \{e\}$. **Thus** $|H * F| = |H||F|/|H \cap F| = p^5$. **Hence** $D = H * F$. **Thus** $D \cong H \times F \cong (Z_{p^4}, +) \times (Z_p, +)$. **Since** $gcd(p, p^4) \neq 1$**, D is not cyclic. It is clear that** D **is abelian.**

QUESTION 6. Let $S = \{0, 1, 2, 3, ..., 8\}$. Then we view S_9 as the set of all bijective functions from S ONTO S, and recall that (S_9, o) is a group. Let $D = \{f : (Z_9, +) \to (Z_9, +) \mid f \ \ is\ a\ group-isomorphism\}$. Hence $D \subset S_9$.

(i) **(8 points)**. Let $K : (Z_9, +) \to (Z_9, +)$ such that $k(1) = 1^8 = 8$. Is $K \in D$? EXPLAIN. Find $K(a)$ for every $a \in Z_9$. If $K \in D$, then find $|K|$.

Solution: Note that $K = (1\ 8)(2\ 7)(3\ 6)(4\ 5)$. **By staring at** K**, we observe that** $K(x) = 0$ **iff** $x = 0$. **Thus** $ker(K) = \{0\}$. **Hence** $K \in D$ **and** $|K| = 2$.

(ii) **(8 points)**.Prove that (D, o) is a cyclic subgroups of S_9 with exactly 6 elements. Hence $D =< f >$ for some $f \in D$. Give me such f.

Solution: Let $f(x) \in D$. Since $Z_9 =< 1 >$, we conclude that $f(x)$ is completely determined by $f(1)$. Since f is an isomorphism and $Z_9 =< 1 >$, we conclude that $Z_9 =< f(1) >$. Now, we know that Z_9 has exactly $\phi(9) = 6$ generators. Let G be the set of all generators of Z_9. Then $G = \{1^1 = 1, 1^2 = 2, 1^4 = 4, 1^5 = 5, 1^7 = 7, 1^8 = 8\}$. Thus $f(1)$ has exactly 6 possibilities. Hence D has exactly 6 elements, namely: f_1 determined by $f_1(1) = 1 = e$, f_2 determined by $f_2(1) = 1^2$, f_3 determined by $f_4(1) = 1^4$, f_5 determined by $f_5(1) = 1^5$, f_7 determined by $f_7(1) = 1^7$, and f_8 determined by $f_8(1) = 1^8$.

To show that $(D, 0)$ is a subgroup of S_9, we only show closure. Let $f_i,\ f_j \in D$, for some $i, j \in G = \{1, 2, 4, 5, 7, 8\}$. We show $f_i\ o\ f_j \in D$. Hence $f_j(1) = 1^j$. Thus $(f_i\ o\ f_j)(1) = f_i(f_j(1)) = f_i(1^j) = 1^{ij} \in D$ since $gcd(ij, 9) = 1$ and thus $ij\ (mod\ 9) \in G$. Assume $D =< f_i >$ for some $i \in G$. Then $|f_i| = 6$. Observe that $f_i^6 = f_i\ o\ f_i\ o \cdots o\ f_i\ (6\ times)\ = f_1 = e$. Observe that $|f_2| = 6$ since $f_2^6(1) = 1^{2^6} = 1^{64} = 1$ in Z_9. Hence $D =< f_2 >$.

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