ON 2-ABSORBING PRIMARY IDEALS IN
COMMUTATIVE RINGS

AYMAN BADAWI, UNSAL TEKIR, AND ECE YETKIN

ABSTRACT. Let \(R\) be a commutative ring with \(1 \neq 0\). In this paper, we introduce the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal \(I\) of \(R\) is called a 2-absorbing primary ideal of \(R\) if whenever \(a, b, c \in R\) and \(abc \in I\), then \(ab \in I\) or \(ac \in \sqrt{I}\) or \(bc \in \sqrt{I}\). A number of results concerning 2-absorbing primary ideals and examples of 2-absorbing primary ideals are given.

1. Introduction

We assume throughout this paper that all rings are commutative with \(1 \neq 0\). Let \(R\) be a commutative ring. An ideal \(I\) of \(R\) is said to be proper if \(I \neq R\). Let \(I\) be a proper ideal of \(R\). Then \(Z_I(R) = \{r \in R \mid rs \in I \text{ for some } s \in R \setminus I\}\). The concept of 2-absorbing ideal, which is a generalization of prime ideal, was introduced by Badawi in [3] and studied in [2], [8], and [4]. Various generalizations of prime ideals are also studied in [1] and [5]. Recall that a proper ideal \(I\) of \(R\) is called a 2-absorbing ideal of \(R\) if whenever \(a, b, c \in R\) and \(abc \in I\), then \(ab \in I\) or \(ac \in \sqrt{I}\) or \(bc \in \sqrt{I}\). In this paper, we introduce the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal \(I\) of \(R\) is said to be a 2-absorbing primary ideal of \(R\) if whenever \(a, b, c \in R\) with \(abc \in I\), then \(ab \in I\) or \(ac \in \sqrt{I}\) or \(bc \in \sqrt{I}\).

Note that a 2-absorbing ideal of a commutative ring \(R\) is a 2-absorbing primary ideal of \(R\). However, these are different concepts. For instance, consider the ideal \(I = (12)\) of \(\mathbb{Z}\). Since \(2 \cdot 2 \cdot 3 \in I\), but \(2 \cdot 2 \notin I\) and \(2 \cdot 3 \notin I\), \(I\) is not a 2-absorbing ideal of \(\mathbb{Z}\). However, it is clear that \(I\) is a 2-absorbing primary ideal of \(\mathbb{Z}\). It is also clear that every primary ideal of a ring \(R\) is a 2-absorbing primary ideal of \(R\). However, the converse is not true. For example, \((6)\) is a 2-absorbing primary ideal of \(\mathbb{Z}\), but it is not a primary ideal of \(\mathbb{Z}\).

Among many results in this paper, it is shown (Theorem 2.2) that the radical of a 2-absorbing primary ideal of a ring \(R\) is a 2-absorbing ideal of \(R\). It is shown (Theorem 2.4) that if \(I_1\) is a \(P_1\)-primary ideal of \(R\) for some prime ideal
$P_1$ of $R$ and $I_2$ is a $P_2$-primary ideal of $R$ for some prime ideal $P_2$ of $R$, then $I_1 I_2$ and $I_1 \cap I_2$ are 2-absorbing primary ideals of $R$. It is shown (Theorem 2.8) that if $I$ is a proper ideal of a ring $R$ such that $\sqrt{I}$ is a prime ideal of $R$, then $I$ is a 2-absorbing primary ideal of $R$. It is shown (Theorem 2.10) that every proper ideal of a divided ring is a 2-absorbing primary ideal. It is shown (Theorem 2.11) that a Noetherian domain $R$ is a Dedekind domain if and only if a nonzero 2-absorbing primary ideal of $R$ is either $M_k$ for some maximal ideal $M$ of $R$ and some positive integer $k \geq 1$ or $M_1^k M_2^n$ for some distinct maximal ideals $M_1, M_2$ of $R$ and some positive integers $k, n \geq 1$. It is shown (Theorem 2.19) that a proper ideal $I$ of $R$ is a 2-absorbing primary ideal if and only if whenever $I_1 I_2 I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $R$, then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq \sqrt{I}$ or $I_2 I_3 \subseteq \sqrt{I}$. Let $R = R_1 \times R_2$, where $R_1, R_2$ are commutative rings with $1 \neq 0$. It is shown (Theorem 2.23) that a proper ideal $J$ of $R$ is a 2-absorbing primary ideal of $R$ if and only if either $J = I_1 \times I_2$ for some 2-absorbing primary ideal $I_1$ of $R_1$ or $J = R_1 \times I_2$ for some 2-absorbing primary ideal $I_2$ of $R_2$ or $J = I_1 \times I_2$ for some primary ideal $I_1$ of $R_1$ and some primary ideal $I_2$ of $R_2$.  

2. Properties of 2-absorbing primary ideals

**Definition 2.1.** A proper ideal $I$ of $R$ is called a 2-absorbing primary ideal of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

**Theorem 2.2.** If $I$ is a 2-absorbing primary ideal of $R$, then $\sqrt{I}$ is a 2-absorbing primary ideal of $R$.

**Proof.** Let $a, b, c \in R$ such that $abc \in \sqrt{I}$, $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$. Since $abc \in \sqrt{I}$, there exists a positive integer $n$ such that $(abc)^n = a^n b^n c^n \in I$. Since $J$ is 2-absorbing primary and $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$, we conclude that $a^n b^n = (ab)^n \in I$, and hence $ab \in \sqrt{I}$. Thus $\sqrt{I}$ is a 2-absorbing ideal of $R$. \[\square\]

**Theorem 2.3.** Suppose that $I$ is a 2-absorbing primary ideal of $R$. Then one of the following statements must hold.

1. $\sqrt{I} = P$ is a prime ideal,
2. $\sqrt{I} = P_1 \cap P_2$, where $P_1$ and $P_2$ are the only distinct prime ideals of $R$ that are minimal over $I$.

**Proof.** Suppose that $I$ is a 2-absorbing primary ideal of $R$. Then $\sqrt{I}$ is a 2-absorbing ideal by Theorem 2.2. Since $\sqrt{\sqrt{I}} = \sqrt{I}$, the claim follows from [3, Theorem 2.4]. \[\square\]

**Theorem 2.4.** Let $R$ be a commutative ring with $1 \neq 0$. Suppose that $I_1$ is a $P_1$-primary ideal of $R$ for some prime ideal $P_1$ of $R$, and $I_2$ is a $P_2$-primary ideal of $R$ for some prime ideal $P_2$ of $R$. Then the following statements hold.

1. $I_1 I_2$ is a 2-absorbing primary ideal of $R$.
2. $I_1 \cap I_2$ is a 2-absorbing primary ideal of $R$. 


Proof. (1) Suppose that \( abc \in I_1I_2 \) for some \( a,b,c \in R \), \( ac \notin \sqrt{I_1I_2} \), and \( bc \notin \sqrt{I_1I_2} = P_1 \cap P_2 \). Then \( a,b,c \notin \sqrt{I_1I_2} = P_1 \cap P_2 \). Since \( \sqrt{I_1I_2} = P_1 \cap P_2 \), we conclude that \( \sqrt{I_1I_2} \) is a 2-absorbing ideal of \( R \). Since \( \sqrt{I_1I_2} \) is a 2-absorbing ideal of \( R \) and \( ac, bc \notin \sqrt{I_1I_2} \), we have \( ab \in \sqrt{I_1I_2} \). We show that \( ab \in I_1I_2 \). Since \( ab \in \sqrt{I_1I_2} \subseteq P_1 \), we may assume that \( a \in P_1 \). Since \( a \notin \sqrt{I_1I_2} \) and \( ab \notin \sqrt{I_1I_2} \subseteq P_2 \), we conclude that \( a \notin P_2 \) and \( b \in P_2 \). Since \( b \in P_2 \) and \( b \notin \sqrt{I_1I_2} \), we have \( b \notin P_1 \). If \( a \in I_1 \) and \( b \in I_2 \), then \( ab \in I_1I_2 \) and we are done. Thus assume that \( a \notin I_1 \). Since \( I_1 \) is a \( P_1 \)-primary ideal of \( R \) and \( a \notin I_1 \), we have \( ab \in P_1 \). Since \( b \in P_2 \) and \( be \in P_1 \), we have \( be \in \sqrt{I_1I_2} \), which is a contradiction. Thus \( a \in I_1 \). Similarly, assume that \( b \notin I_2 \). Since \( I_2 \) is a \( P_2 \)-primary ideal of \( R \) and \( b \notin I_2 \), we have \( ac \notin P_2 \). Since \( ac \in P_2 \) and \( a \in P_1 \), we have \( ac \in \sqrt{I_1I_2} \), which is a contradiction. Thus \( b \in I_2 \). Hence \( ab \in I_1I_2 \). 

(2)(Similar to the proof in (1)). Let \( H = I_1 \cap I_2 \). Then \( \sqrt{H} = P_1 \cap P_2 \). Suppose that \( abc \in H \) for some \( a,b,c \in R \), \( ac \notin \sqrt{H} \), and \( bc \notin \sqrt{H} \). Then \( a,b,c \notin \sqrt{H} = P_1 \cap P_2 \). Since \( \sqrt{H} = P_1 \cap P_2 \) is a 2-absorbing ideal of \( R \) and \( ac, bc \notin \sqrt{H} \), \( ab \in \sqrt{H} \). We show that \( ab \in H \). Since \( ab \in \sqrt{H} \subseteq P_1 \), we may assume that \( a \in P_1 \). Since \( a \notin \sqrt{H} \) and \( ab \notin \sqrt{H} \subseteq P_2 \), we conclude that \( a \notin P_2 \) and \( b \in P_2 \). Since \( b \in P_2 \) and \( b \notin \sqrt{H} \), \( b \notin P_1 \). If \( a \in I_1 \) and \( b \in I_2 \), then \( ab \notin H \) and we are done. Thus assume that \( a \notin I_1 \). Since \( I_1 \) is a \( P_1 \)-primary ideal of \( R \) and \( a \notin I_1 \), we have \( bc \in P_1 \). Since \( b \in P_2 \) and \( bc \in P_1 \), we have \( bc \in \sqrt{H} \), which is a contradiction. Thus \( a \in I_1 \). Similarly, assume that \( b \notin I_2 \). Since \( I_2 \) is a \( P_2 \)-primary ideal of \( R \) and \( b \notin I_2 \), we have \( ac \notin P_2 \). Since \( ac \in P_2 \) and \( a \in P_1 \), we have \( ac \in \sqrt{H} \), which is a contradiction. Thus \( b \in I_2 \). Hence \( ab \in H \). \( \square \)

In view of Theorem 2.4, we have the following result.

**Corollary 2.5.** Let \( R \) be a commutative ring with \( 1 \neq 0 \), and let \( P_1, P_2 \) be prime ideals of \( R \). If \( P_1^n \) is a \( P_1 \)-primary ideal of \( R \) for some positive integer \( n \geq 1 \) and \( P_2^n \) is a \( P_2 \)-primary ideal of \( R \) for some positive integer \( m \geq 1 \), then \( P_1^n P_2^m \) and \( P_1^n \cap P_2^m \) are 2-absorbing primary ideals of \( R \). In particular, \( P_1P_2 \) is a 2-absorbing primary ideal of \( R \).

In the following example, we show that if \( P_1, P_2 \) are prime ideals of a ring \( R \) and \( n, m \) are positive integers, then \( P_1^n P_2^m \) need not be a 2-absorbing primary ideal of \( R \).

**Example 2.6.** Let \( R = \mathbb{Z}[Y] + 3XZ[3Y, X] \). Then \( P_1 = YR \) and \( P_2 = 3XZ[3Y, X] \) are prime ideals of \( R \). Let \( I = P_1 P_2 \). Then \( 3X^2 \cdot Y \cdot 3 = 9X^2Y \in I \) and \( 3X^2 \cdot Y = 3X^2Y \notin I \). Clearly \( 3X^2 \cdot 3 = 9X^2 \notin \sqrt{I} = P_1 \cap P_2 \) and \( Y \cdot 3 = 3Y \notin \sqrt{I} = P_1 \cap P_2 \). Hence \( I \) is not a 2-absorbing primary ideal of \( R \).

In the following example, we show that if \( I \subseteq J \) such that \( I \) is a 2-absorbing primary ideal of \( R \) and \( \sqrt{J} = \sqrt{I} \), then \( J \) need not be a 2-absorbing ideal of \( R \).

**Example 2.7.** Let \( R = \mathbb{Z}[X, Y, Z] \). Then \( P_1 = XR, P_2 = YR \) are prime ideals of \( R \), and \( I = P_1 P_2 \) is a 2-absorbing primary ideal of \( R \) by Corollary 2.5. Let
Let $I$ be a proper ideal of a ring $R$. It is known that if $\sqrt{J}$ is a maximal ideal of $R$, then $I$ is a prime ideal of $R$. In the following result, we show that if $\sqrt{J}$ is a prime ideal of $R$, then $I$ is a 2-absorbing primary ideal of $R$.

**Theorem 2.8.** Let $I$ be an ideal of $R$. If $\sqrt{J}$ is a prime ideal of $R$, then $I$ is a 2-absorbing primary ideal of $R$. In particular, if $P$ is a prime ideal of $R$, then $P^n$ is a 2-absorbing primary ideal of $R$ for every positive integer $n \geq 1$.

**Proof.** Suppose that $abc \in I$ and $ab \notin I$. Since $(ac)(bc) = abc^2 \in I \subseteq \sqrt{J}$ and $\sqrt{J}$ is a prime ideal of $R$, we have $bc \in \sqrt{J}$ or $ac \in \sqrt{J}$. Hence $I$ is a 2-absorbing primary ideal of $R$. \hfill $\square$

In view of Theorem 2.2, Theorem 2.3, and Theorem 2.8, the following is an example of an ideal $J$ of a ring $R$ where $\sqrt{J}$ is a 2-absorbing ideal of $R$, but $J$ is not a 2-absorbing primary ideal of $R$.

**Example 2.9.** Let $R = \mathbb{Z}[X, Y, Z]$ and let $J = (XYZ, Y^3, X^3)R$. Then $\sqrt{J} = XR \cap YR$ is a 2-absorbing ideal of $R$, but $J$ is not a 2-absorbing primary ideal of $R$ by Example 2.7. Also, see Example 2.6.

Recall that a commutative ring $R$ with $1 \neq 0$ is called a divided ring if for every prime ideal $P$ of $R$, we have $P \subseteq xR$ for every $x \in R \setminus P$. Every chained ring is a divided ring (recall that a commutative ring $R$ with $1 \neq 0$ is called a chained ring, if $x | y$ in $R$ or $y | x$ in $R$ for every $x, y \in R$). It is known that the prime ideals of a divided ring are linearly ordered; i.e., if $P_1, P_2$ are prime ideals of $R$, then $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. We have the following result.

**Theorem 2.10.** Let $R$ be a commutative divided ring with $1 \neq 0$. Then every proper ideal of $R$ is a 2-absorbing primary ideal of $R$. In particular, every proper ideal of a chained ring is a 2-absorbing primary ideal.

**Proof.** Let $I$ be a proper ideal of $R$. Since the prime ideals of a divided ring are linearly ordered, we conclude that $\sqrt{J}$ is a prime ideal of $R$. Hence $I$ is a 2-absorbing primary ideal of $R$ by Theorem 2.8. \hfill $\square$

Let $R$ be an integral domain with $1 \neq 0$, and let $K$ be the quotient field of $R$. If $I$ is a nonzero proper ideal of $R$, then $I^{-1} = \{ x \in K \mid xI \in R \}$. An integral domain $R$ is said to be a Dedekind domain if $II^{-1} = R$ for every nonzero proper ideal $I$ of $R$.

**Theorem 2.11.** Let $R$ be a Noetherian integral domain with $1 \neq 0$ that is not a field. Then the following statements are equivalent.

1. $R$ is a Dedekind domain.
(2) A nonzero proper ideal $I$ of $R$ is a 2-absorbing primary ideal of $R$ if and only if either $I = M^n$ for some maximal ideal $M$ of $R$ and some positive integer $n \geq 1$ or $I = M^1_1 M^2_2$ for some maximal ideals $M_1, M_2$ of $R$ and some positive integers $n, m \geq 1$.

(3) If $I$ is a nonzero proper 2-absorbing primary ideal of $R$, then either $I = M^n$ for some maximal ideal $M$ of $R$ and some positive integer $n \geq 1$ or $I = M^1_1 M^2_2$ for some maximal ideals $M_1, M_2$ of $R$ and some positive integers $n, m \geq 1$.

(4) A nonzero proper ideal $I$ of $R$ is a 2-absorbing primary ideal of $R$ if and only if either $I = P^n$ for some prime ideal $P$ of $R$ and some positive integer $n \geq 1$ or $I = P^1 P^2 m$ for some prime ideals $P_1, P_2$ of $R$ and some positive integers $n, m \geq 1$.

(5) If $I$ is a nonzero proper 2-absorbing primary ideal of $P_1, P_2$ of $R$ and some positive integers $n, m \geq 1$.

Proof. (1) $\Rightarrow$ (2). Suppose that $R$ is a Dedekind domain that is not a field. Then every nonzero prime ideal of $R$ is maximal. Let $I$ be a nonzero proper ideal of $R$. Then $I = M^n_1 \cdots M^n_k$ for some distinct maximal ideals $M_1, \ldots, M_k$ of $R$ and some positive integers $n_1, \ldots, n_k \geq 1$. Suppose that $I$ is a 2-absorbing primary ideal of $R$. Since every nonzero prime ideal of $R$ is maximal and $\sqrt{I}$ is either a maximal ideal of $R$ or $I_1 \cap I_2$ for some maximal ideals $I_1, I_2$ of $R$ by Theorem 2.3, we conclude that either $I = M_n$ for some maximal ideal $M$ of $R$ and some positive integer $n \geq 1$ or $I = M^1_1 M^2_2$ for some maximal ideals $M_1, M_2$ of $R$ and some positive integers $n, m \geq 1$. Conversely, suppose that $I = M^n$ for some maximal ideal $M$ of $R$ and some positive integer $n \geq 1$ or $I = M^1_1 M^2_2$ for some maximal ideals $M_1, M_2$ of $R$ and some positive integers $n, m \geq 1$. Then $I$ is a 2-absorbing primary ideal of $R$ by Theorem 2.8 and Corollary 2.5.

(2) $\Rightarrow$ (3). It is clear.

(2) $\Rightarrow$ (4). It is clear.

(4) $\Rightarrow$ (5). It is clear.

(3) $\Rightarrow$ (5). It is clear.

(5) $\Rightarrow$ (1). Let $M$ be a maximal ideal of $R$. Since every ideal between $M^2$ and $M$ is an $M$-primary ideal, and hence a 2-absorbing primary ideal of $R$, the hypothesis in (5) implies that there are no ideals properly between $M^2$ and $M$. Hence $R$ is a Dedekind domain by [6, Theorem 39.2, p. 470].

Since every principal ideal domain is a Dedekind domain, we have the following result as a consequence of Theorem 2.11.

**Corollary 2.12.** Let $R$ be a principal ideal domain and $I$ be a nonzero proper ideal of $R$. Then $I$ is a 2-absorbing primary ideal of $R$ if and only if either $I = p^n R$ for some prime element $p$ of $R$ and $k \geq 1$ or $I = p^n_1 p^n_2 R$ for some...
distinct prime elements $p_1, p_2$ of $R$ and some positive integers $n, m \geq 1$. In particular, if $R = \mathbb{Z}$ or $R = F[X]$ for some field $F$, then a proper ideal $I$ of $R$ is a 2-absorbing primary ideal of $R$ if and only if $I = p^kR$ for some prime element $p$ of $R$ and some positive integer $k \geq 1$ or $I = p_1^n p_2^m R$ for some distinct prime elements $p_1, p_2$ of $R$ and some positive integers $n, m \geq 1$.

The following is an example of a unique factorization domain that contains a 2-absorbing primary ideal not of the form $P_1^n P_2^m$ for some prime ideals $P_1, P_2$ of $R$ and some positive integers $n, m \geq 1$.

**Example 2.13.** Let $R = K[X, Y]$, where $K$ is a field. Consider the ideal $I = (X, Y^2)$ of $R$. Then $I$ is a 2-absorbing primary ideal of $R$ that is not of the form $P_1^n P_2^m$, where $P_1, P_2$ are prime ideals of $R$ and $n, m \geq 1$.

Let $R$ be a commutative Noetherian ring with $1 \neq 0$. It is well-known that every proper ideal of $R$ has a primary decomposition. Since every primary ideal is a 2-absorbing primary ideal, we conclude that every proper ideal of $R$ has a 2-absorbing primary decomposition. However, decomposition of an ideal of $R$ into 2-absorbing primary ideals need not be unique. We have the following example.

**Example 2.14.** In light of Corollary 2.12, consider the ideal $(60)$ of $\mathbb{Z}$. Then $(60) = (3) \cap (4) \cap (5) = (3) \cap (20) = (4) \cap (15) = (5) \cap (12)$.

Hence $(60)$ has four distinct 2-absorbing primary decompositions. The ideal $(210)$ of $\mathbb{Z}$ has exactly ten distinct 2-absorbing primary decompositions.

$(210) = (2) \cap (3) \cap (5) \cap (7) = (6) \cap (5) \cap (7) = (10) \cap (3) \cap (7) = (14) \cap (3) \cap (5) = (15) \cap (2) \cap (7) = (15) \cap (14) = (21) \cap (2) \cap (5) = (21) \cap (10) = (35) \cap (2) \cap (3) = (35) \cap (6)$.

**Definition 2.15.** Let $I$ be a 2-absorbing primary ideal of $R$. Then $P = \sqrt{I}$ is a 2-absorbing ideal by Theorem 2.2. We say that $I$ is a $P$-2-absorbing primary ideal of $R$.

**Theorem 2.16.** Let $I_1, I_2, \ldots, I_n$ be $P$-2-absorbing primary ideals of $R$ for some 2-absorbing ideal $P$ of $R$. Then $I = \bigcap_{i=1}^n I_i$ is a $P$-2-absorbing primary ideal of $R$.

**Proof.** First observe that $\sqrt{I} = \bigcap_{i=1}^n \sqrt{I_i} = P$. Suppose that $abc \in I$ for some $a, b, c \in R$ and $ab \notin I$. Then $ab \notin I_i$ for some $1 \leq i \leq n$. Hence $bc \in \sqrt{I_i} = P$ or $ac \in \sqrt{I_i} = P$. 

If $I_1, I_2$ are 2-absorbing primary ideals of a ring $R$, then $I_1 \cap I_2$ need not be a 2-absorbing primary ideal of $R$. We have the following example.

**Example 2.17.** Let $I_1 = 50\mathbb{Z}$ and $I_2 = 75\mathbb{Z}$. Then $I_1, I_2$ are 2-absorbing primary ideals of $\mathbb{Z}$ by Corollary 2.12. Since $\sqrt{I_1} \cap \sqrt{I_2} = 2\mathbb{Z} \cap 3\mathbb{Z} \cap 5\mathbb{Z} = 30\mathbb{Z}$, $I_1 \cap I_2$ is not a 2-absorbing primary ideal of $\mathbb{Z}$ by Theorem 2.3.
In the following result, we show that a proper ideal $I$ of a ring $R$ is a 2-absorbing primary ideal of $R$ if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $R$, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$. But first we have the following lemma.

**Lemma 2.18.** Let $I$ be a 2-absorbing primary ideal of a ring $R$ and suppose that $abJ \subseteq I$ for some elements $a, b \in R$ and some ideal $J$ of $R$. If $ab \notin I$, then $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$.

**Proof.** Suppose that $aJ \notin \sqrt{I}$ and $bJ \notin \sqrt{I}$. Then $a_j \notin \sqrt{I}$ and $b_j \notin \sqrt{I}$ for some $j_1, j_2 \in J$. Since $a_j \notin \sqrt{I}$ and $b_j \notin \sqrt{I}$, we have $bj \in \sqrt{I}$. Since $abj \in I$ and $ab \notin I$ and $aj \notin \sqrt{I}$, we have $b \notin \sqrt{I}$. Now, since $ab(\overline{j}_1 + \overline{j}_2) \in I$ and $ab \notin I$, we have $a(\overline{j}_1 + \overline{j}_2) \in \sqrt{I}$ or $b(\overline{j}_1 + \overline{j}_2) \in \sqrt{I}$. Suppose that $a(\overline{j}_1 + \overline{j}_2) = a_j + a_j \notin \sqrt{I}$. Since $a_j \notin \sqrt{I}$, we have $a \notin \sqrt{I}$, a contradiction. Suppose that $b(\overline{j}_1 + \overline{j}_2) = b_j + b_j \notin \sqrt{I}$. Since $b_j \notin \sqrt{I}$, we have $b \notin \sqrt{I}$, a contradiction again. Thus $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$.

**Theorem 2.19.** Let $I$ be a proper ideal of $R$. Then $I$ is a 2-absorbing primary ideal if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $R$, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$.

**Proof.** Suppose that whenever $I_1I_2I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $R$, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$. Then clearly $I$ is a 2-absorbing primary ideal of $R$ by definition.

Conversely, suppose that $I$ is a 2-absorbing primary ideal of $R$ and $I_1I_2I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $R$, such that $I_1I_2 \not\subseteq I$. We show that $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$. Suppose that neither $I_1I_3 \subseteq \sqrt{I}$ nor $I_2I_3 \subseteq \sqrt{I}$. Then there are $q_1 \in I_1$ and $q_2 \in I_2$ such that neither $q_1I_3 \subseteq \sqrt{I}$ nor $q_2I_3 \subseteq \sqrt{I}$. Since $q_1q_2I_3 \subseteq I$ and neither $I_1I_3 \subseteq \sqrt{I}$ nor $q_2I_3 \subseteq \sqrt{I}$, we have $I_1q_2 \in I$ by Lemma 2.18.

Since $I_1I_2 \not\subseteq I$, we have $ab \notin I$ for some $a \in I_1, b \in I_2$. Since $abI_3 \subseteq I$ and $ab \notin I$, we have $aI_3 \subseteq \sqrt{I}$ or $bI_3 \subseteq \sqrt{I}$ by Lemma 2.18. We consider three cases. **Case one:** Suppose that $aI_3 \subseteq \sqrt{I}$ but $bI_3 \subseteq \sqrt{I}$. Since $q_1bI_3 \subseteq I$ and neither $bl_3 \subseteq \sqrt{I}$ nor $q_1l_3 \subseteq \sqrt{I}$, we conclude that $q_1b \in I$ by Lemma 2.18. Since $(a + q_1)bI_3 \subseteq I$ and $aI_3 \subseteq \sqrt{I}$, but $q_1I_3 \subseteq \sqrt{I}$, we conclude that $(a + q_1)I_3 \subseteq \sqrt{I}$. Since neither $bl_3 \subseteq \sqrt{I}$ nor $(a + q_1)I_3 \subseteq \sqrt{I}$, we conclude that $(a + q_1)b \in I$ by Lemma 2.18. Since $(a + q_1)b = ab + q_1b \in I$ and $q_1b \notin I$, we conclude that $ab \in I$, a contradiction. **Case two:** Suppose that $bI_3 \subseteq \sqrt{I}$, but $aI_3 \not\subseteq \sqrt{I}$. Since $aq_2I_3 \subseteq I$ and neither $aI_3 \subseteq \sqrt{I}$ nor $q_2I_3 \subseteq \sqrt{I}$, we conclude that $aq_2 \in I$. Since $ab(q_2)I_3 \subseteq I$ and $bl_3 \subseteq \sqrt{I}$, but $q_2I_3 \not\subseteq \sqrt{I}$, we conclude that $(b + q_2)I_3 \not\subseteq \sqrt{I}$. Since neither $aI_3 \subseteq \sqrt{I}$ nor $(b + q_2)I_3 \subseteq \sqrt{I}$, we conclude that $a(b + q_2) \in I$ by Lemma 2.18. Since $a(b + q_2) = ab + aq_2 \in I$ and $aq_2 \in I$, we conclude that $ab \in I$, a contradiction. **Case three:** Suppose that $aI_3 \subseteq \sqrt{I}$ and $bl_3 \subseteq \sqrt{I}$. Since $bI_3 \subseteq \sqrt{I}$ and $q_2I_3 \not\subseteq \sqrt{I}$, we conclude that $(b + q_2)I_3 \not\subseteq \sqrt{I}$. Since $q_1(b + q_2)I_3 \subseteq I$ and neither $q_1I_3 \subseteq \sqrt{I}$ nor
Let \( q_1I_3 \subseteq \sqrt{T} \), we conclude that \( q_1(b + q_2) = q_1b + q_1q_2 \in I \) by Lemma 2.18. Since \( q_1q_2 \subseteq I \) and \( q_1b + q_1q_2 \subseteq I \), we conclude that \( bq_1 \in I \). Since \( aI_3 \subseteq \sqrt{T} \) and \( q_1I_3 \not\subseteq \sqrt{T} \), we conclude that \( (a+q_1)I_3 \not\subseteq \sqrt{T} \). Since \( (a+q_1)q_2I_3 \subseteq I \) and neither \( q_2I_3 \subseteq \sqrt{T} \) nor \( (a+q_1)I_3 \not\subseteq \sqrt{T} \), we conclude that \( (a+q_1)q_2 = aq_2 + q_1q_2 \in I \) by Lemma 2.18. Since \( q_1q_2 \subseteq I \) and \( aq_2 + q_1q_2 \subseteq I \), we conclude that \( aq_2 \in I \). Now, since \( (a + q_1)(b + q_2)I_3 \subseteq I \) and neither \( (a + q_1)I_3 \not\subseteq \sqrt{T} \) nor \( (b + q_2)I_3 \not\subseteq \sqrt{T} \), we conclude that \( (a + q_1)(b + q_2) = ab + aq_2 + bq_1 + q_1q_2 \in I \) by Lemma 2.18. Since \( aq_2, bq_1, q_1q_2 \subseteq I \), we have \( aq_2 + bq_1 + q_1q_2 \subseteq I \). Since \( ab + aq_2 + bq_1 + q_1q_2 \subseteq I \) and \( aq_2 + bq_1 + q_1q_2 \subseteq I \), we conclude that \( ab \in I \), a contradiction. Hence \( I_1I_3 \subseteq \sqrt{T} \) or \( I_2I_3 \subseteq \sqrt{T} \). \( \square \)

**Theorem 2.20.** Let \( f : R \to R' \) be a homomorphism of commutative rings. Then the following statements hold.

1. If \( I' \) is a 2-absorbing primary ideal of \( R' \), then \( f^{-1}(I') \) is a 2-absorbing primary ideal of \( R \).

2. If \( f \) is an epimorphism and \( I \) is a 2-absorbing primary ideal of \( R \) containing \( \ker f \), then \( f(I) \) is a 2-absorbing primary ideal of \( R' \).

**Proof.** (1) Let \( a, b, c \in R \) such that \( abc \in f^{-1}(I') \). Then \( f(abc) = f(a)f(b)f(c) \in I' \). Hence we have \( f(a)f(b) \in I' \) or \( f(b)f(c) \in \sqrt{T} \) or \( f(a)f(c) \in \sqrt{T} \), and thus \( ab \in f^{-1}(I') \) or \( ac \in f^{-1}(\sqrt{T}) \) or \( bc \in f^{-1}(\sqrt{T}) \). By using the equality \( f^{-1}(\sqrt{T}) = \sqrt{f^{-1}(T)} \), we conclude that \( f^{-1}(I') \) is a 2-absorbing primary ideal of \( R \).

(2) Let \( a', b', c' \in R' \) and \( a'b'c' \in f(I) \). Then there exist \( a, b, c \in R \) such that \( f(a) = a', f(b) = b', f(c) = c' \), and \( f(abc) = a'b'c' \in f(I) \). Since \( \ker f \subseteq I \), we have \( abc \in I \). It implies that \( ab \in I \) or \( ac \in \sqrt{T} \) or \( bc \in \sqrt{T} \). This means that \( a'b' \in f(I) \) or \( a'c' \in f(\sqrt{T}) \subseteq \sqrt{f(I)} \) or \( b'c' \in f(\sqrt{T}) \subseteq \sqrt{f(I)} \). Thus \( f(I) \) is a 2-absorbing primary ideal of \( R' \). \( \square \)

**Corollary 2.21.** Let \( R \) be a commutative ring with \( 1 \neq 0 \). Suppose that \( I, J \) are distinct proper ideals of \( R \). If \( J \subseteq I \) and \( I \) is a 2-absorbing primary ideal of \( R \), then \( I/J \) is a 2-absorbing primary ideal of \( R/J \).

**Proof.** The proof is clear by Theorem 2.20(2). \( \square \)

**Theorem 2.22.** Let \( R \) be a commutative ring with \( 1 \neq 0 \), \( S \) be a multiplicatively closed subset of \( R \), and \( I \) be a proper ideal of \( R \). Then the following statements hold.

1. If \( I \) is a 2-absorbing primary ideal of \( R \) such that \( I \cap S = \emptyset \), then \( S^{-1}I \) is a 2-absorbing primary ideal of \( S^{-1}R \).

2. If \( S^{-1}I \) is a 2-absorbing primary ideal of \( S^{-1}R \) and \( S \cap Z_I(R) = \emptyset \), then \( I \) is a 2-absorbing primary ideal of \( R \).

**Proof.** (1) Let \( a, b, c \in R \), \( s, t, k \in S \) such that \( \frac{a t b}{s t k} \in S^{-1}I \). Then there exists \( u \in S \) such that \( uabc \in I \). Since \( I \) is a 2-absorbing primary ideal, we get
Then \( \frac{ab}{c} \in I \) or \( bc \in \sqrt{I} \) or \( uac \in \sqrt{I} \). If \( uab \in I \), then \( \frac{ab}{c} = \frac{uab}{uc} \in S^{-1}I \). If \( bc \in \sqrt{I} \), then \( \frac{ab}{c} = \frac{uab}{uc} \in S^{-1}I \). If \( uac \in \sqrt{I} \), then \( \frac{ab}{c} = \frac{uab}{uc} \in \sqrt{S^{-1}I} \).

(2) Let \( a, b, c \in R \) such that \( abc \in I \). Then \( \frac{abc}{c} = \frac{a}{c} \leq \frac{a}{1} \in S^{-1}I \). It follows \( \frac{a}{c} \in S^{-1}I \) or \( \frac{ab}{1} \in \sqrt{S^{-1}I} \) or \( \frac{ac}{1} \in \sqrt{S^{-1}I} \). If \( \frac{a}{c} = \frac{ac}{c} \), then \( uab \in I \), for some \( u \in S \). Since \( u \in S \) and \( S \cap Z(I(R)) = \emptyset \), we conclude \( ab \in I \) if \( \frac{a}{c} = \frac{ac}{c} \in \sqrt{S^{-1}I} \), then there exists \( v \in S \) and a positive integer \( n \) such that \( (vbc)^n = v^n b^c c^n \in I \). Since \( v \in S \), we have \( v^n \not\in Z(I(R)) \). Thus \( b^n c^n \in I \), and so \( bc \in \sqrt{I} \). If \( \frac{a}{c} \in \sqrt{S^{-1}I} \), then similarly we obtain \( ac \in \sqrt{I} \), and it completes the proof. \( \square \)

**Theorem 2.23.** Let \( R = R_1 \times R_2 \), where \( R_1 \) and \( R_2 \) are commutative rings with \( 1 \neq 0 \). Let \( J \) be a proper ideal of \( R \). Then the following statements are equivalent.

1. \( J \) is a 2-absorbing primary ideal of \( R \).
2. Either \( J = I_1 \times I_2 \) for some 2-absorbing primary ideal \( I_1 \) of \( R_1 \) or \( J = I_1 \times I_2 \) for some 2-absorbing primary ideal \( I_2 \) of \( R_2 \) or \( J = I_1 \times I_2 \) for some primary ideal \( I_1 \) of \( R_1 \) and some primary ideal \( I_2 \) of \( R_2 \).

**Proof.** (1) \( \Rightarrow \) (2). Assume that \( J \) is a 2-absorbing primary ideal of \( R \). Then \( J = I_1 \times I_2 \) for some ideal \( I_1 \) of \( R_1 \) and some ideal \( I_2 \) of \( R_2 \). Suppose that \( I_2 = R_2 \). Since \( I_2 \) is a proper ideal of \( R \), \( I_1 \neq R_1 \). Let \( J' = \frac{J}{\langle 0 \rangle \times R_2} \) is a 2-absorbing primary ideal of \( J' \) by Corollary 2.21. Since \( R' \) is ring-isomorphic to \( R_1 \) and \( I_1 \cong J' \), then \( I_1 \) is a 2-absorbing primary ideal of \( R_1 \). Suppose that \( I_1 \neq R_1 \). Since \( J \) is a proper ideal of \( R \), \( I_2 \neq R_2 \). By a similar argument as in the previous case, \( I_2 \) is a 2-absorbing primary ideal of \( R_2 \). Hence assume that \( I_1 \neq R_1 \) and \( I_2 \neq R_2 \). Then \( \sqrt{J} = \sqrt{I_1} \times \sqrt{I_2} \). Suppose that \( I_1 \) is not a primary ideal of \( R_1 \). Then there are \( a, b \in R_1 \) such that \( ab \in I_1 \) but neither \( a \in I_1 \) nor \( b \in I_1 \). Let \( x = (a, 1), y = (1, 0) \), and \( c = (b, 1) \). Then \( yzc = (ab, 0) \) in \( J \) but neither \( xy = (a, 0) \) in \( J \) nor \( yc = (b, 0) \) in \( \sqrt{J} \), which is a contradiction. Thus \( I_1 \) is a primary ideal of \( R_1 \). Suppose that \( I_2 \) is not a primary ideal of \( R_2 \). Then there are \( d, e \in R_2 \) such that \( de \in I_2 \) but neither \( d \in I_2 \) nor \( e \in \sqrt{I_2} \). Let \( x = (1, d), y = (0, 1) \), and \( c = (1, e) \). Then \( yzc = (0, de) \) in \( J \) but neither \( xy = (0, d) \) in \( J \) nor \( xc = (1, de) \) in \( \sqrt{J} \), which is a contradiction. Thus \( I_2 \) is a primary ideal of \( R_2 \).

(2) \( \Rightarrow \) (1). If \( J = I_1 \times R_2 \) for some 2-absorbing primary ideal \( I_1 \) of \( R_1 \) or \( J = R_1 \times I_2 \) for some 2-absorbing primary ideal \( I_2 \) of \( R_2 \), then it is clear that \( J \) is a 2-absorbing primary ideal of \( R \). Hence assume that \( J = I_1 \times I_2 \) for some primary ideal \( I_1 \) of \( R_1 \) and some primary ideal \( I_2 \) of \( R_2 \). Then \( I_1' = I_1 \times I_2 \) and \( I_2' = R_1 \times I_2 \) are primary ideals of \( R \). Hence \( I_1' \cap I_2' = I_1 \times I_2 = J \) is a 2-absorbing primary ideal of \( R \) by Theorem 2.4. \( \square \)

**Theorem 2.24.** Let \( R = R_1 \times R_2 \times \cdots \times R_n \), where \( 2 \leq n < \infty \), and \( R_1, R_2, \ldots, R_n \) are commutative rings with \( 1 \neq 0 \). Let \( J \) be a proper ideal of \( R \). Then the following statements are equivalent.
(1) $J$ is a 2-absorbing primary ideal of $R$.
(2) Either $J = \times_{t=1}^n I_t$ such that for some $k \in \{1, 2, \ldots, n\}$, $I_k$ is a 2-absorbing primary ideal of $R_k$, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k\}$ or $J = \times_{t=1}^n I_t$ such that for some $k, m \in \{1, 2, \ldots, n\}$, $I_k$ is a primary ideal of $R_k$, $I_m$ is a primary ideal of $R_m$, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k, m\}$.

**Proof.** We use induction on $n$. Assume that $n = 2$. Then the result is valid by Theorem 2.23. Thus let $3 \leq n < \infty$ and assume that the result is valid when $K = R_1 \times \cdots \times R_{n-1}$. We prove the result when $R = K \times R_n$. By Theorem 2.23, $J$ is a 2-absorbing primary ideal of $R$ if and only if either $J = L \times R_n$ for some 2-absorbing primary ideal $L$ of $K$ or $J = K \times L_n$ for some 2-absorbing primary ideal $L_n$ of $R_n$. Observe that a proper ideal $Q$ of $K$ is a primary ideal of $K$ if and only if $Q = \times_{t=1}^{n-1} I_t$ such that for some $k \in \{1, 2, \ldots, n-1\}$, $I_k$ is a primary ideal of $R_k$ and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n-1\} \setminus \{k\}$. Thus the claim is now verified. 

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**References**


Ayman Badawi  
Department of Mathematics & Statistics  
American University of Sharjah  
P.O. Box 26666, Sharjah, United Arab Emirates  
E-mail address: abadawi@aus.edu

Unsal Tekir  
Department of Mathematics  
Marmara University  
İstanbul, Turkey  
E-mail address: utekir@marmara.edu.tr
Ece Yetkin
Department of Mathematics
Marmara University
Istanbul, Turkey
E-mail address: ece.yetkin@marmara.edu.tr