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Department of Mathematics and Statistics
American University of Sharjah

Final Exam – Spring 2022
MTH 221 – Linear Algebra

Date: Saturday March 21, 2022 Time: 5:00-7:00 pm.

Student Name	Student ID Number
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1. This exam has 8 page plus this cover page.
2. *No communication of any kind!*
3. *Do not open this exam until you are told to begin.*
4. *No Questions are allowed during the examination.*
5. *Do not separate the pages of the exam.*
6. *Scientific calculators are allowed but cannot be shared. Graphing Calculators are not allowed.*
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Student signature: ______________________

Name Lilian, ID 82168

Final Exam, MTH 221 , Spring 2022

Ayman Badawi

Score = $\frac{}{53}$

(35 points) Section A (Written Questions): Show All Your Work. No credit will be given if work is NOT shown.

(i) (3 points) Show that the set $W = \left\{ \begin{bmatrix} a & 3a \\ b & 0 \end{bmatrix} \mid a, b \in R \right\}$ is a subspace of $R^{2\times 2}$.

$W = \{ a\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mid a,b \in R \}$

Span $= \{ \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \}$

can be written as span → subspace of $R^{2\times 2}$

(ii) (3 points) Let $T : P_3 \to P_3$ be a linear transformation such that

$$T(a_2x^2 + a_1x + a_0) = (a_2 + 3a_1)x^2 + (a_1 - a_0)x + (a_1 + 3a_0)$$

Find the eigenvalues of T.(Note the eigenvalues of T are the eigenvalues of the co-matrix presentation of the co-linear transformation of T).

find comatrix first. $(a_2(x^2) + a_1(3x^2 + x + 1) + a_0(-x+3))$, a_2, a_1, a_0

$$\begin{array}{cc} & \begin{array}{ccc} a_2 & a_1 & a_0 \end{array} \\ A = & \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} \end{array}$$

To find eigenvalues : $[\alpha I_3 - A] = 0$

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \alpha-1 & -3 & 0 \\ 0 & \alpha-1 & 1 \\ 0 & -1 & \alpha-3 \end{bmatrix}$$

Determinant method : $(\alpha-1)(-1)^{1+1}[(\alpha-1)(\alpha-3)+1]$

$(\alpha-1)[(\alpha-1)(\alpha-3)+1]$

$(\alpha-1)[\alpha^2 - 4\alpha + 3 + 1] = (\alpha-1)(\alpha^2 - 4\alpha + 4)$

$\alpha^3 - 4\alpha^2 + 4\alpha - \alpha^2 + 4\alpha - 4 = 0$

$= \alpha^3 - 5\alpha^2 + 8\alpha - 4 = 0$

$(\alpha-1)(\alpha-2)^2$

eigenvalues: $\alpha = 1$ once, $\alpha = 2$ twice repeated.

(iii) (6 points) Let $T : R^5 \to R^4$ be an R-homomorphism (i.e., linear transformation) such that

$$T(a_1, a_2, a_3, a_4, a_5) = (a_1 + 3a_3 - 2a_5,\ 2a_1 + 6a_3 - 2a_5,\ 3a_1 + 9a_3 - 6a_5,\ -3a_1 - 9a_3 - a_4 + 6a_5)$$

a. (0.5 point) Find the standard matrix presentation, M, of T.

$$\underset{4\times5}{M} = \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 3 & 0 & -2 \\ 2 & 0 & 6 & 0 & -2 \\ 3 & 0 & 9 & 0 & -6 \\ -3 & 0 & -9 & -1 & 6 \end{bmatrix} \end{array}$$

b. (2 points) Find a basis for the column space of M.

Row operations.

$-2R_1 + R_2 \to R_2$
$-3R_1 + R_3 \to R_3$
$3R_1 + R_4 \to R_4$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \quad \begin{array}{c} -1R_4 \\ \frac{1}{2}R_2 \end{array} \quad \begin{bmatrix} 1 & 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Basis = $\{(1,0,3,0,-2), (0,0,0,0,1), (0,0,0,1,0)\}$

Basis for $Col(M) = \{(1,2,3,-3), (0,0,0,-1), (-2,-2,-6,6)\}$

3 independent, $\dim(T) = 3$

c. (0.5 point) Find a basis for the range of T.

Basis = $\{(1,2,3,-3), (0,0,0,-1), (-2,-2,-6,6)\}$

$Range(T) = Col(M)$

d. (2 points) Find all points in the domain of T such that $T(a_1, a_2, a_3, a_4, a_5) = (1, 2, 3, 5)$.

augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & -2 & 1 \\ 2 & 0 & 6 & 0 & -2 & 2 \\ 3 & 0 & 9 & 0 & -6 & 3 \\ -3 & 0 & -9 & -1 & 6 & 5 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 8 \end{array}\right]$$

$-1R_4$
$\frac{1}{2}R_2$
$-2R_2 + R_1 \to R_1$

completely reduced matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -8 \end{array}\right]$$

Read

$a_1 + 3a_3 = 1$, $a_1 = 1 - 3a_3$
$a_5 = 0$ free variables: a_3, a_2
$a_4 = -8$ leading: a_1, a_4, a_5

$S.S = \{(1-3a_3, a_2, a_3, -8, 0) \mid a_2, a_3 \in R\}$

not subspace.

e. (1 point) Find a basis for the $Z(T) = Null(T) = Ker(T)$.

$S.S = \{(-3a_3, a_2, a_3, 0, 0) \mid a_2, a_3 \in R\}$

$S.S = \{a_3(-3,0,1,0,0) + a_2(0,1,0,0,0) \mid a_2, a_3 \in R\}$

Basis = $\{(-3,0,1,0,0), (0,1,0,0,0)\}$

(iv) (5 points) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 8 \end{bmatrix}$. It is clear that 2, 2, 8 are the eigenvalues of A. lower triangular matrix

a. (3 points) Find a basis for the eigenspace E_2.

$$\begin{bmatrix} \alpha-2 & 0 & 0 \\ 0 & \alpha-2 & 0 \\ -3 & 0 & \alpha-8 \end{bmatrix}$$

$$E_2 \to \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 0 & -6 & 0 \end{array}\right] \overset{-\frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

Read $x_1 + 2x_3 = 0$, $x_1 = -2x_3$

$S.S = \{(-2x_3, x_2, x_3) \mid x_2, x_3 \in R\}$

$S.S = \{x_3(-2,0,1) + x_2(0,1,0) \mid x_2, x_3 \in R\}$

Basis $= \{(-2,0,1), (0,1,0)\}$

span $= \{(-2,0,1), (0,1,0)\}$

b. To know if diagonalizable, find E_8

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{array}\right] \overset{\frac{1}{6}R_1,\ \frac{1}{6}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{array}\right] \overset{3R_1+R_3 \to R_3}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Read $x_1 = 0$, $x_2 = 0$, $x_3 \to$ free variable.

$S.S = \{(0,0,x_3) \mid x_3 \in R\}$

$\{x_3(0,0,1) \mid x_3 \in R\}$

span $= \{0,0,1\}$

b. (2 points) Is A diagonalizable? Justify your answer.

Yes

$E_2 \to \dim(E_2) = 2 = n_2 = 2$

$E_8 \to \dim(E_8) = 1 = n_8 = 1$

It is diagonalizable

$(\alpha-2)^{2}(\alpha-8)^{1} = 0$

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $Q = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(v) (3 points) Let

$$A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$ upper triangular matrix

Find a diagonal matrix D and an invertible matrix Q such that $D = Q^{-1}AQ$. You don't need to find Q^{-1}

$(\alpha I_2 - A)$

$$\begin{bmatrix} \alpha-1 & 2 \\ 0 & \alpha+1 \end{bmatrix}$$

$(\alpha-1)(\alpha+1) = 0$

eiganvalues $\rightarrow \alpha = 1, \alpha = -1$

Find eigan spaces

$E_1 \rightarrow \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 2 & 0 \end{array}\right] \overset{\frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \overset{-2R_1+R_2 \rightarrow R_2}{\sim} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = 0$ $x_1 \rightarrow$ free variable

$S.S = \{(x_1, 0) \mid x_1 \in \mathbb{R}\} \sim \{x_1(1,0) \mid x_1 \in \mathbb{R}\}$

span $= \{(1,0)\}$

$E_{-1} \rightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \overset{-\frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 - x_2 = 0, \quad x_1 = x_2$ leading $\rightarrow x_1$
free variable $\rightarrow x_2$

$S.S = \{(x_2, x_2) \mid x_2 \in \mathbb{R}\} \rightarrow \{x_2(1,1) \mid x_2 \in \mathbb{R}\}$

span $= \{(1,1)\}$

diagnolizable ✓

$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(vi) (4 points) Let $T : R^2 \to R^2$ be a linear transformation such that $T(4,0) = (12,4)$ and $T(-4,1) = (-10,-3)$

a. (2 points) Find the standard matrix presentation of T.

$$\frac{1}{4}T(4,0) = \frac{1}{4}(12,4)$$
$$T(1,0) = (3,1)$$
$$T(4,0) + T(-4,1) = T(0,1)$$
$$(12,4) + (-10,-3) = (2,1)$$
$$T(0,1) = (2,1)$$
$$M = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

b. (2 points) I claim that T^{-1} exists. Find $T^{-1}(2,6)$ $|A| = 3\times1 - 2\times1 = 1$

$$M^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \sim \quad T = M^{-1}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underset{2\times2}{\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 2 \\ 6 \end{bmatrix}}$$
$$= 2\begin{vmatrix} 1 \\ -1 \end{vmatrix} + 6\begin{vmatrix} -2 \\ 3 \end{vmatrix} = \underset{2\times1}{\begin{bmatrix} -10 \\ 16 \end{bmatrix}}$$

(vii) (4 points)

a. (2 points) Convince me that $D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + d = 1 \right\}$ is not a subspace of $R^{2\times2}$

$d = 1 - a$

$$D = \left\{ \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \middle| \, a,b,c \in R \right\}$$ not a subspace

If $a, b, c = 0$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin D$ $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in D$

b. (2 points) Given that $D = \{(a-2b)x^3 + (-2a+4b)x^2 + (3a-6b)x + 2a - 4b \mid a, b \in R\}$ is a subspace of P_3. Write D as span of independent polynomials.

$$D = \{ a(x^3 - 2x^2 + 3x + 2) + b(-2x^3 + 4x^2 - 6x - 4) \mid a,b \in R \}$$

$$\text{Span} = \{ (x^3 - 2x^2 + 3x + 2), (-2x^3 + 4x^2 - 6x - 4) \}$$

dependent

(viii) (7 points) Consider the mapping $T: P_3 \to R^2$ defined by $T(ax^2+bx+c) = (a, b+c)$

a. (2 points) Convince me that T is a linear transformation.

$T(a,b,c) = \{a(1,0) + b(0,1) + c(0,1) \mid a,b,c \in \mathbb{R}\}$

It is linear transformation because $a, b+c$ are linear combinations of a,b,c

b. (3 points) Find a basis for $Z(T) = \ker(T) = Null(T)$.

$L: R^3 \to R^2$

$$\underset{2\times 3}{M} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$a = 0$, $b + c = 0$, $b = -c$

$a, b \to$ leading

$c \to$ free variable

$L(a,b,c) = \{(0, -c, c) \mid a,b,c \in \mathbb{R}\}$

$L(a,b,c) = (0, -1, 1)$

$T: P_3 \to R^2$

$T(ax^2+bx+c) = -x + 1$

Basis $= \{(-x+1)\}$

✓ $\dim(Z(T)) = 1$ (no. of free variables)

c. (2 point) Is T onto? Justify your answer.

✓

onto $\to$ Range (T) = co-domain

we know: $\dim(\text{Range}) + \dim(Z(T))$ = domain

$\dim(\text{Range}) + 1 = 3$

$\dim(\text{Range}) = 2$ = co-domain

So T is onto.

(i) (18 points): Section B (MCQ): All your answers for the MCQ must be included in the below table.

Question	Answer
1	a
2	a
3	b
4	b
5	d
6	d
7	c
8	d
9	c
Total/18	

(i) The angle between the polynomial $f_1(x) = x$ and the polynomial $f_2(x) = 4x^2$ in P_3 with respect to the inner product $< f(x), g(x) > = \int_0^1 f(x)g(x)\,dx$ is

$$\int_0^1 x \times 4x^2\,dx = \int_0^1 4x^3\,dx = \frac{4x^4}{4}\Big|_0^1 = x^4\Big|_0^1 = 1$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{\int_0^1 x^2\,dx}\sqrt{\int_0^1 16x^4\,dx}}\right)$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{\frac{1}{3}}\sqrt{\frac{16}{5}}}\right) = 14.47^\circ$$

a. 14.47 degrees
b. 0.25 degrees
c. 20.36 degrees
d. 75.99 degrees

(ii) Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 8 \end{bmatrix}$. Which of the following statements is always correct?

a. A is diagonalizable
b. A is invertible
c. A is not diagonalizable +
d. $\lambda = -2$ is an eigenvalue ×

(iii) Let $T : R^9 \to R^7$ be a linear transformation such that $dim(Range) = 4$. Then $dim(Z(T) = dim(Ker(T) = dim(Null(T)) =$

dim(Range) + dim(Z(T)) = domain
4 + ? = 9
dim(Z(T)) = 5

a. 3
b. 5
c. 4
d. 6

(iv) Let $B = span\,\{u_1 = (1,3,0,0), u_2 = (2,6,-1,0), u_3 = (2,0,-1,0)\}$, where $\{u_1, u_2, u_3\}$ is a basis for D. Then Gram-Schmidt process can be applied to transform $B = \{u_1, u_2, u_3\}$ into an orthogonal basis $O = \{w_1, w_2, w_3\}$ for D. Use the standard dot product on D to find the vector w_2 in the basis O. Then w_2 is

$w_2 = u_2 - \frac{u_2 \cdot w_1}{|w_1|^2} \cdot w_1$

$= (2,6,-1,0) - \frac{(1,3,0,0)\cdot(2,6,-1,0)}{1^2+3^2} \times (1,3,0,0)$

$= (2,6,-1,0) - \frac{20}{10}(1,3$

a. $(1,0,0,0)$
b. $(0,0,-1,0)$
c. $(0,-2,1,0)$
d. $(1,0,-2,0)$

(v) Let $v_1 = (1,-1,0)$ and $v_2 = (0,2,0)$. Given $D = \{Q = (a,b,c) \in R^3 \mid Q$ is orthogonal to both v_1 and $v_2\}$ is a subspace of R^3 (assume the normal dot product on D). A basis for D is

$\begin{matrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{matrix}$

a. $\{(0,-1,2)\}$
b. $\{(1,0,2)\}$
c. $\{(0,0,2),(1,0,1)\}$
d. $\{(0,0,2)\}$

(vi) Let A be a 3×3 matrix such that $C_A(\alpha) = (\alpha-1)^2(\alpha-5)$, where $E_1 = span\{(1,1,1),(-2,-2,0)\}$ and $E_5 = span\{(-3,0,-3)$. Given $D = A^2 + 5A^{-1} + 2I_3$. Then Trace(D) is $\alpha = 1$, $\alpha = 5$

eigenvalues of D $= 1^2 + \frac{5}{1} + 2 = 8$ twice

D $= 5^2 + \frac{5}{5} + 2 = 28$

Trace(D) = 8 + 8 + 28

a. 7
b. 6
c. 36
d. 44

(vii) Let $T : R^2 \to P_3$ be a linear transformation defined by $T(a,b) = ax^2 + bx$. Which of the following statements is always correct?

a(1,0,0) + b(0,1,0)
span: (1,0,0), (0,1,0) range
dim(Range) = 2
domain = 2
dim(Z(T)) = 0
not onto
1-1 ✓
not isomorphis

a. T is an onto linear transformation
b. T is not a one to one linear transformation
c. T is not an onto linear transformation
d. T is an isomorphism (i.e., one to one and onto)

(viii) Let $A = (-9,4)$, $B = (-4,6)$ and $C = (1,12)$. The area of the triangle ABC is

(a) 20 (b) 4 (c) 40 (d) = 10

$v_1 = AB = (5,2)$
$v_2 = AC = (10,8)$
Area $= \frac{\begin{vmatrix} 5 & 2 \\ 10 & 8 \end{vmatrix}}{2} = \frac{20}{2} = 10$

(ix) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 5 & 3 & a_4 \\ 4 & 2 & a_5 \end{bmatrix}$ and $B = \begin{bmatrix} a_1 & a_2 & a_3+2 \\ 5 & 3 & a_4 \\ 4 & 2 & a_5 \end{bmatrix}$. Assume $|A| = 20$. Then $|B| =$

(a) 22 (b) 18 (c) 16 (d) = 24

Faculty information

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