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# Exam One, MTH 205, Spring 2022

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Score = 50 v. good / 50

**QUESTION 1. (7 points)** (more clearly written on the back of last page)

see the last page for clear details

Given $f(t) = \begin{cases} 2 & if\ 0 \le t < 2 \\ 0 & if\ t \ge 2 \end{cases}$

Solve for $y(t)$ where

$$y'' - 3y' = f(t), \quad y(0) = y'(0) = 0$$

$f(t) = 2[U_0(t) - U_2(t)] + 0[U_2(t)]$

$f(t) = 2[1 - U(t-2)]$

$y'' - 3y' = 2[1 - U(t-2)]$

apply laplace

$s^2Y(s) - 3sY(s) = 2\mathcal{L}\{1 - U(t-2)\}$

$s^2Y(s) - 3sY(s) = \frac{2}{s} - 2\frac{e^{-2s}}{s}$

$Y(s)(s^2 - 3s) = \frac{2 - 2e^{-2s}}{s}$

$Y(s) = 2\frac{1 - e^{-2s}}{s(s^2 - 3s)} = 2\frac{1 - e^{-2s}}{s^2(s-3)}$

$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-3)} - \frac{e^{-2s}}{s^2(s-3)}\right\}$

$= 2\left[\mathcal{L}^{-1}\left\{-\frac{1}{9s} - \frac{1}{3s^2} + \frac{1}{9(s-3)}\right\} - \mathcal{L}^{-1}\{e^{-2s}Y(s)\}\right]$

$= 2\left[-\frac{1}{9} - \frac{1}{3}t + \frac{1}{9}e^{3t} - \mathcal{L}^{-1}\{e^{-2s}Y(s)\}\right]$

$\frac{1}{s^2(s-3)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-3}$

$b = -\frac{1}{3} \quad c = \frac{1}{9}$

$as(s-3) + b(s-3) + cs^2 = 1$

$as^2 - 3as + bs - 3b + cs^2 = 1$

$a + c = 0 \quad -3a + b = 0$

$-3b = 1 \quad -3a = \frac{1}{3}$

$a = -\frac{1}{9}$

$2\left[-\frac{1}{9} - \frac{1}{3}t + \frac{1}{9}e^{3t}\right.$

$\left.-U_2(t)\left(-\frac{1}{9} - \frac{1}{3}(t-2) + \frac{1}{9}e^{3(t-2)}\right)\right]$

$y(t) = 2\left[-\frac{1}{9} - \frac{1}{3}t + \frac{1}{9}e^{3t} + \frac{1}{9}U_2(t) + \frac{1}{3}U_2(t)(t-2) - \frac{1}{9}U_2(t)e^{3(t-2)}\right]$

7

**QUESTION 2. (7 points)**
Solve for $y(t)$ where

$$y'' - 4y' + 5y = \delta_2(t), \quad y(0) = y'(0) = 0$$

apply laplace

$s^2Y(s) - 4sY(s) + 5Y(s) = e^{-2s}$

$Y(s)(s^2 - 4s + 5) = e^{-2s}$

$Y(s) = \frac{e^{-2s}}{s^2 - 4s + 5} = \frac{e^{-2s}}{(s-2)^2 + 1}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s-2)^2+1}\right\} = \boxed{1e^{2(t-2)}\sin(t-2)\,U(t-2)}$

$F(s) = \frac{1}{(s-2)^2+1}$

$f(t) = e^{2t}\sin t \qquad f(t-2) = e^{2(t-2)}\sin(t-2)$

7

**QUESTION 3. (7 points)** Solve for $y(t)$ where

$$y' - 6y = 1 - 9\int_{r=0}^{r=t} y(r)\,dr,\ y(0) = 0$$

apply laplace

$$sY(s) - 6Y(s) = \frac{1}{s} - 9\,\mathcal{L}\left\{\int_{r=0}^{r=t} y(r)\,dr\right\}$$

$$sY(s) - 6Y(s) = \frac{1}{s} - \frac{9Y(s)}{s}$$

$$sY(s) - 6Y(s) + \frac{9}{s}Y(s) = \frac{1}{s}$$

$$Y(s)\left(s - 6 + \frac{9}{s}\right) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s-6+9/s)} = \frac{1}{s^2 - 6s + 9} = \frac{1}{(s-3)^2}$$

$$y(t) = te^{3t}$$

7

**QUESTION 4. (6 points)** Solve for $x(t)$ ONLY. Given $x(0) = 0, x'(0) = x''(0) = 0, y(0) = y'(0) = 0$

$$x''(t) - y'(t) = 0$$

$$x^{(3)}(t) + y''(t) = 48t + 12$$

[Note $\ell\{x^{(3)}(t)\} = s^3X(S) - s^2x(0) - sx'(0) - x''(0)$ ]

apply laplace

$$s^2X(s) - sY(s) = 0$$

$$s^3Y(s) + s^2Y(s) = \frac{48}{s^2} + \frac{12}{s}$$

$$\frac{48 + 12s}{s^2} =$$

cramer

$$X(s) = \frac{\begin{vmatrix} 0 & -s \\ \frac{48+12s}{s^2} & s^2 \end{vmatrix}}{\begin{vmatrix} s^2 & -s \\ s^3 & s^2 \end{vmatrix}} = \frac{\frac{48+12s}{s}}{s^4 + s^4} = \frac{48+12s}{s(2s^4)} = \frac{48+12s}{2s^5} = \frac{24+6s}{s^5}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{24}{s^5} + \frac{6}{s^4}\right\}$$

$$= t^4 + t^3$$

6

**QUESTION 5. (10 points)** Find the general solution of $y(t)$, i.e., find $y_g$ where

$$ty'' - 2y' + \frac{2y}{t} = \frac{1}{t}$$

$y_h$ Cauchy set $y = t^n$ $y' = nt^{n-1}$ $y'' = (n^2-n)t^{n-2}$

$t\, t^{n-2}(n^2-n) - 2n\,t^{n-1} + \frac{2t^n}{t} = 0$

$t^{n-1}(n^2-n) - 2n\,t^{n-1} + 2t^{n-1} = 0$

$t^{n-1}(n^2 - 3n + 2) = 0$

$n = 2$ $y_1 = t^2$

$n = 1$ $y_2 = t$

$y_h = c_1 t^2 + c_2 t$

find $y_p$ by variation $= v_1 y_1 + v_2 y_2$

$\frac{1}{t} \div t$

$\frac{1}{t} \cdot \frac{1}{t} = \frac{1}{t^2}$

$v_1' t^2 + v_2' t = 0$

$2v_1' t + v_2' = \frac{1}{t^2}$

$$v_1' = \frac{\begin{vmatrix} 0 & t \\ 1/t^2 & 1 \end{vmatrix}}{\begin{vmatrix} t^2 & t \\ 2t & 1 \end{vmatrix}} = \frac{-1/t}{t^2 - 2t^2} = \frac{-1/t}{-t^2} = \frac{1}{t^3}$$

$v_1 = \int \frac{1}{t^3}\, dt = \int t^{-3}\, dt = -\frac{t^{-2}}{2} = -\frac{1}{2t^2}$

10

$t v_2' = -t^2 \frac{1}{t^3}$

$t v_2' = -\frac{1}{t}$ $v_2 = -\int t^{-2}\, dt$

$v_2' = -\frac{1}{t^2}$ $= -\frac{t^{-1}}{-1} = \frac{1}{t}$

$y_p = -\frac{1}{2t^2}(t^2) + \frac{1}{t}(t) = -\frac{1}{2} + 1 = \frac{1}{2}$

$y_g(t) = c_1 t^2 + c_2 t + \frac{1}{2}$

**QUESTION 6. (5 points)** Given $y_1 = t$ is a solution to the homogeneous Linear Diff. Equation $y'' - \frac{2t}{1+t^2}y' + a_0(t)y = 0$. Find $y_h(t)$.

$$y_h = C_1 y_1 + C_2 y_2 = C_1 t + C_2(t^2-1)$$

$$y_2 = y_1 \int \frac{e^{-\int \frac{a_1(t)}{a_2(t)} dt}}{y_1^2} dt$$

$$\int \frac{2t}{1+t^2} dt \qquad u = 1+t^2, \quad du = 2t$$

$$\int \frac{2t}{u} \frac{du}{2t} = \int \frac{du}{u} = \ln|1+t^2|$$

$$y_2 = t \int \frac{e^{-\int \frac{-2t}{1+t^2} dt}}{t^2}$$

$$= t \int \frac{e^{\int \frac{2t}{1+t^2} dt}}{t^2} dt$$

$$= t \int \frac{1+t^2}{t^2} dt = t \int \frac{1}{t^2} + 1 \, dt$$

$$= t \int t^{-2} + 1 \, dt$$

$$= t \left[ \frac{t^{-1}}{-1} + t \right] = t\left(-\frac{1}{t} + t\right)$$

$$= -1 + t^2$$

$$= \boxed{t^2 - 1}$$

**QUESTION 7. (8 points)**

(i) **(3 points)** Find $y_h$, where $y'' - 5y' + 6y = 0$

set $y = e^{mt}$

Charac: $m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$

$m = 3 \quad y_1 = e^{3t}$

$m = 2 \quad y_2 = e^{2t}$

$$y_h = C_1 e^{3t} + C_2 e^{2t}$$

3

ii) **(3 points)** Consider the L.D. E $y'' - 5y' + 6y = t^2$. Use Laplace, as explained in the class room, and find the general form of $y_p$.

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}$$

$$Y(s) = \frac{\phantom{2}}{s^3(s-3)(s-2)} = \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{s^3} + \frac{C_4}{s-3} + \frac{C_5}{s-2}$$

$$y(t) = \underbrace{C_1 + C_2 t + C_3 t^2}_{y_p} + \underbrace{C_4 e^{3t} + C_5 e^{2t}}_{y_h}$$

general form $\quad y_p = C_1 + C_2 t + C_3 t^2$

3

(iii) **(2 points)** By substituting $y_p$ in the diff. equation in (ii), find the exact $y_p$.

$y_p' = C_2 + 2C_3 t$

$y_p'' = 2C_3$

$2C_3 - 5(C_2 + 2C_3 t) + 6(C_1 + C_2 t + C_3 t^2) = t^2$

$2C_3 - 5C_2 - 10C_3 t + 6C_1 + 6C_2 t + 6C_3 t^2 = t^2$

2

$6C_3 = 1 \qquad 2C_3 - 5C_2 + 6C_1 = 0$

$C_3 = 1/6 \qquad -10C_3 + 6C_2 = 0$

Faculty information

$-\frac{10}{6} + 6C_2 = 0$

$C_2 = \frac{5}{18}$

$C_1 = \frac{19}{108}$

$y_p = \frac{1}{6}\left(\frac{19}{18} + \frac{5}{3}t + t^2\right) \qquad$ so $\boxed{y_p = \frac{19}{108} + \frac{5}{18}t + \frac{1}{6}t^2}$

question one with more details

$y'' - 3y' = f(t)$

$f(t) = 2[u_0(t) - u_2(t)]$

$= 2u_0(t) - 2u_2(t) = 2 - 2u(t-2)$

apply laplace

$s^2 Y(s) - 3sY(s) = \frac{2}{s} - \frac{2e^{-2s}}{s}$

$Y(s)(s^2 - 3s) = \frac{2}{s} - \frac{2e^{-2s}}{s}$

$Y(s) = \frac{2 - 2e^{-2s}}{s(s^2-3s)} = \frac{2 - 2e^{-2s}}{s^2(s-3)}$

$= \frac{2}{s^2(s-3)} - \frac{2e^{-2s}}{s^2(s-3)}$

$\frac{1}{s^2(s-3)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-3}$

$b = -\frac{1}{3}$ $a = -\frac{1}{9}$ $c = \frac{1}{9}$

$= 2\left(-\frac{1}{9s} - \frac{1}{3s^2} + \frac{1}{9(s-3)}\right)$

$-2e^{-2s}\left(-\frac{1}{9s} - \frac{1}{3s^2} + \frac{1}{9(s-3)}\right)$

$y(t) = 2\mathcal{L}^{-1}\left\{-\frac{1}{9s} - \frac{1}{3s^2} + \frac{1}{9(s-3)}\right\} - 2\mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{-1}{9s} - \frac{1}{3s^2} + \frac{1}{9(s-3)}\right)\right.$

$= 2\left(-\frac{1}{9} - \frac{1}{3}t + \frac{1}{9}e^{3t}\right) - 2u(t-2)\left[-\frac{1}{9} - \frac{1}{3}(t-2) + \frac{1}{9}e^{3(t-2)}\right]$

$= -\frac{2}{9} - \frac{2}{3}t + \frac{2}{9}e^{3t} + \frac{2}{9}u(t-2) + \frac{2}{3}u(t-2)(t-2) - \frac{2}{9}u(t-2)e^{3(t-2)}$

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