MTH205: Review Problems / Final / Spring 08

Q1. For the differential equation: $y' = y/x + \sqrt{x}$, does the differential equation possess a unique solution through the point $(0,0)$? Give reasons.

Q2. Find the critical point and phase portrait of $y' = y^2 - y^3$ and classify each point.

Q3. Classify each of the following differential equations as, separable, exact, linear, homogeneous, or Bernoulli. Then find the general solution of each of them.

a) $2xyy' + y^2 = 2x^2$, b) $y' = xy + \sqrt{y}$, c) $y' = \dfrac{x - 2xy}{3y^2 + x^2}$, d) $y' = e^{3x-2y}$

Q4. Suppose that a large mixing tank initially holds 500 gallons of water in which 50 pounds of salt have been dissolved. Water is pumped into the tank at rate of r_1 gal/min, and then when the solution is well stirred it is pumped out at rate r_2. Determine a differential equation and solve it for the amount $x(t)$ of the salt in the tank at any time t for each of the following cases:

a) $r_1 = r_2 = 4$ and the entering water is pure.

b) $r_1 = 3, \quad r_2 = 2$ and the entering water contains salt with concentration 2 lb/gal.

Q5. Initially 100 milligrams of radioactive substance were present. After 6 hours the mass had decreased by 3%. If the rate of decay is proportional to the amount of the substance present at any time, find the amount remaining after 24 hours.

Q6 Find an interval around $x = 0$ for which the initial value problem

$$\sqrt{x+1}\; y'' + \frac{1}{4 - x^2} y' + y = \sin(x) \quad , \quad y(0) = 1 \quad , \quad y'(0) = 0$$

has a unique solution.

Q7. Find the general solution

a) $x^3 y''' + 6x^2 y'' + 4xy' - 4y = 0$, b) $y^{(4)} + y'' = 0$,

c) $x^2 y'' - 4xy' + 6y = x^3$

Q8. Given that $y_1 = e^x$ is a solution of the d.e. $(x+1)y'' - (x+2)y' + y = 0$, find the general solution.

Q9. Set up the appropriate form of the particular solution y_p, but do not determine the values of the coefficients.

$$y^{(4)} + 9y'' = 2x + (x^2 + 1)\sin(3x)$$

Q10. Use the method of undetermined coefficients to find the solution of the I.V.P.

$$y'' + 4y = 2\sin(2t) \quad , \quad y(0) = 1, y'(0) = 0$$

Q11. Use the variational method to find the general solution of the d.e.

$$x^2 y'' - 3xy' + 4y = x^2 \ln x$$

Q12. A 32 – pound weight stretches a spring 32/5 feet. Initially the weight is released 1 foot below the equilibrium position with a downward velocity of 5 ft/sec. Determine the equation of the motion, if the surrounding medium offer a damping force numerically equal to 2 times the instantaneous velocity and the weight is driven by an external force equal to $f(t) = 12\cos 2t + 3\sin 2t$. Graph the transient and the steady-state solutions on the same coordinate axes.

Q13. A 64 – pound weight stretches a spring 0.32 foot. Initially the weight is released 2/3 foot above the equilibrium position with a downward velocity of 5 ft/sec.

a) Determine the equation of the motion.
b) What are the amplitude and the period of the motion?
c) At what time does the mass pass through the equilibrium position for the second time?
d) At what time does the mass attain its extreme displacement for the second time?

Q14. Determine a lower bound for the radius of convergence of the series solutions of the d.e. $(x+1)(x^2+2)y''-xy'+2y=0$ about $x=0$.

Q15. Find the first six terms of a series solution about $x=0$ for the IVP

$$(x^2+2)y''-y=0,\ \ y(0)=2,\ \ y'(0)=3$$

Q16. Use translation and other theorems to find :

a) $L\ \{e^{at}\sin t\ \}$ b) $L^{-1}\left\{\dfrac{s}{(s+1)^3}\right\}$ c) $L\ \{g(t)\},\ g(t)=\begin{cases}0\ ,\ 0\le t<1\\ t^2\ ,\quad t\ge 1\end{cases}$

d) $L^{-1}\ \{\dfrac{e^{-2s}}{s^3}\}$ e) $L^{-1}\ \{t^3\cos 5t\}$

Q17. Use the Laplace transform to solve the following initial value problems

a) $y'+4y=e^{-4t}\ ,\quad y(0)=2$

b) $y''+y=f(t),\quad y(0)=0,\quad y'(0)=1$, where $f(t)=\begin{cases}0\ ,\quad 0\le t<\pi\\ 1,\quad \pi\le t<2\pi\\ 0\quad ,\quad t\ge 2\pi\end{cases}$

c) $y''-4y=e^t\quad ,\quad y(0)=1, y'(0)=0$

d) $y''+y=\delta(t-\pi),\ \ y(0)=1, y'(0)=0$

Q18. Use the Convolution Theorim to

a) evaluate $L\ \{\ t^2 * te^t\ \}$, $L^{-1}\left\{\dfrac{1}{s^3(s-1)}\right\}$

b) solve $y'=1-\sin t-\int_0^t y(\tau)\ d\tau,\quad y(0)=0$

Q19. Solve $y''+2y+10y=f(t),\ y(0)=1,\ \ y'(0)=0,$ where f is periodic with period $T=2\pi$ and $f(t)=\begin{cases}1\ \ ,\ 0\le t<\pi\\ -1\ \ ,\quad t\ge\pi\end{cases}$

Q20. Use Laplace Transforms to solve the system

$$x'=2y+e^t,\ \ x(0)=1$$
$$y'=8x-t,\ \ y(0)=1$$