AMERICAN UNIVERSITY of SHARJAH

Differential Equations
MTH 205

Class Notes

Spring 2009
Dr. Ayman Badawi

Subject: (Date / / :التاريخ) :الموضوع

25/1/2009 Sunday

Linear Diff Equation :-

• Def :- The definite integration over positive part of function give the Area but the integration over negative part does not give the Area "in other word we have to use absolute value since area can't be negative".

Ex:- $y' = \frac{-1}{x^2}$ → Linear Diff Equation as long as y' is Not raised to any Power except Power one.

$3y'' + 6y' - 5y = 3x^{10}$ ⇒ Linear Diff Equation of Order 2

$3y'' + 6y' - 5y^2 = 3x^{10}$ ⇒ Not Linear Diff Equation.

Note :-

$y^2 \neq y^{(2)}$

y^2 means y raised to Power 2

$y^{(2)}$ means 2nd Derivative of y

$y^{(4)} + 6y^{(2)} - 5y' = 0$ Linear Diff Equ of Order ~~2~~ 4

* To make solution unique we have to be given number of conditions that equal the number of order.

Subject: (Date / / التاريخ:)

27/1/2009

Tuesda

Def :- $a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \cdots + a_1(x)\,y' + a_0(x)\,y = k$
is a linear Diff Equation of order $\underline{\underline{n}}$.

- Here we like to find y, i.e, our solution should be
$y = \underset{\text{in terms of } x}{\text{.........}}$

→ Here y is dependent variable.
x is Independent variable.

- Linear because y and all the given derivative of y
all to the power one.

- $K(x), a_0(x), a_1(x), \ldots, a_n(x)$ are just functions in ter

Example :-

$$\underbrace{3x}_{a_3(x)}\,y^{(3)} + \underbrace{\frac{1}{x+2}}_{a_1(x)}\,y' + \underbrace{7}_{a_0(x)}\,y = \underbrace{3x^2 - 2}_{K(x)}$$

→ y is dependent
→ x " Independent
→ Linear Diff Equation of order 3

Ex:-

$$3y\, W^{(5)} + (6y+7)\, W^{(3)} + 7W' = y^3 + 2y$$

Linear D.E of order 5

W is Dependent

y is Independent

Our Solution: $W = \underbrace{\text{------}}_{\text{interms of } y}$

Chapter Four :- Theorum:-

leading coffecient

$$\boxed{a_n(x)}\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_1(x)\, y' + a_0(x)\, y = K(x), \quad (*)$$

Suppose that B an interval I such that $K(x), a_0(x), a_1(x), \dots a_n(x)$ are all continuous on I and $a_n(x) \neq 0 \ \forall\, x \in I$

Also, suppose that $a \in I$ and $y(a), y'(a), y^{(2)}(a), \dots\dots y^{(n-1)}(a)$ are determined. Then $a_n(x)$ or $(*)$ has a unique solution on I

Example:-

$$(x^2-1)\, y^{(3)} + \frac{1}{x+4}\, y' + 7y = 2x+1, \quad y(2) = 1, \quad y'(2) = 3, \quad y^{(2)} = 10$$

Find the largest interval that guarantee the Uniquness of our sol

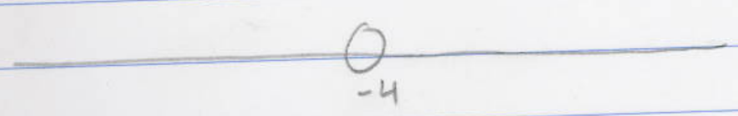
① Linear D.E of order 3

x^2-1 is Cont on $\mathbb{R}$

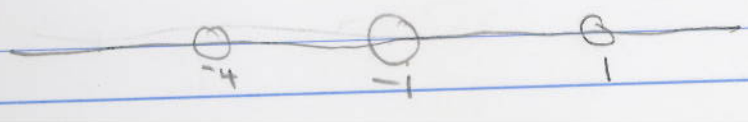
$1/x+4$ " " " $\mathbb{R} \setminus \{-4\}$

7 " " " $\mathbb{R}$

$2x+1$ " " " $\mathbb{R}$



General Domain



Domain after making $a_n \neq 0$

Subject: (Date / / :التاريخ) :موضوع

Continue of the Solution:-

from the given conditions:-

$y(2) = 1 \quad y'(2) = 3 \quad y^{(2)}(2) = 10$

$\therefore \; a = 2$

we will choose $(1, \infty)$ since $a = 2 \in (1, \infty)$ only.

$I = (1, \infty) \rightsquigarrow$ represents the value of x.

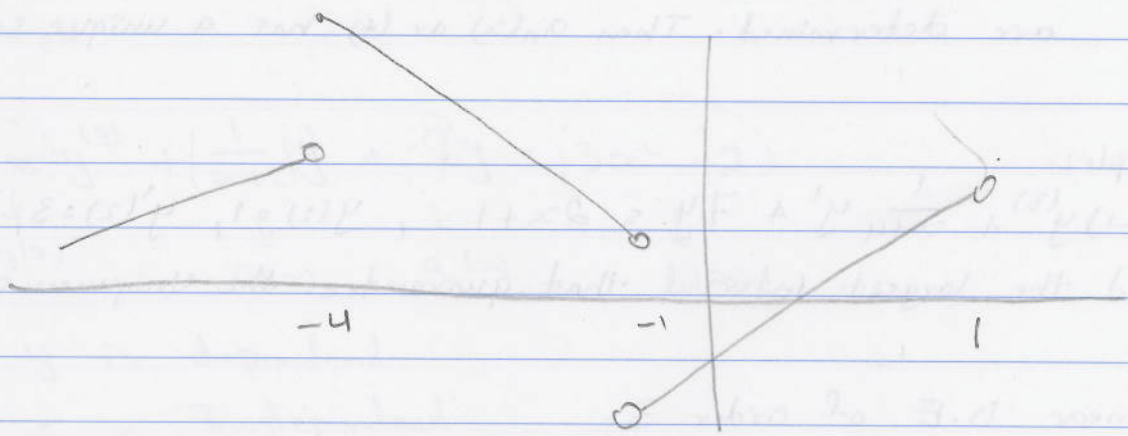
$=$

Note:-

If "a" was changing in the condition, then the theorum will not be valid. Ex:- $a = 2$ then $a = -2$

$=$

In case we pick more than one interval, the graph will



From the graph, the gaps are not accepted in the reality. So we pick only one solution.

$=$

Example:-

$$3t\,w^{(2)} + \frac{1}{t-7}\,w' + \sqrt{t+2}\,w = t^2 + 7, \quad w(1) = w'(1) = 7$$

Find the Largest interval about t so that the solution is Unique

Solution:-

Linear & from order (2)

Dependent = w

Independent = t

Our solution = w = ______ interms of t

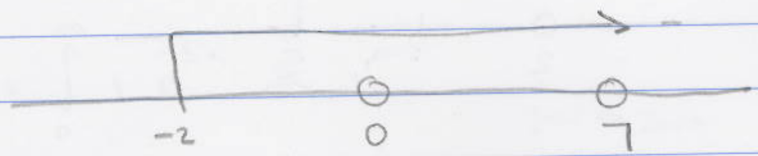
$3t$ is Cont on $\mathbb{R}$

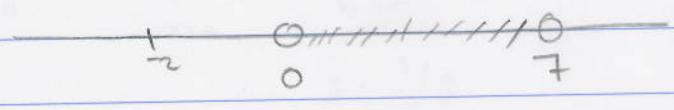
$\frac{1}{t-7}$ " " " $\mathbb{R} \setminus \{7\}$

$\sqrt{t+2}$ " " " $[-2, \infty)$

t^2+7 " " " $\mathbb{R}$

$a = 1$

① 

② 

Unique Solution on $(0,7)$ " value of t must be between 0 & 7 and both of them is Not included.

Subject: (Date / / :التاريخ) :موضوع

Def:-

* Diff Equation without Condition is just Diff Equ

* " " with Condition is Intial Value

H.W:-

$$\sqrt{5-t}\; z^{(3)} + \frac{1}{t^2-4} z' + t z = e^t + 3,$$

$z(3)=10 \quad z'(3)=2 \quad z''(3)=6$

Find the largest interval about t so that solution is unique

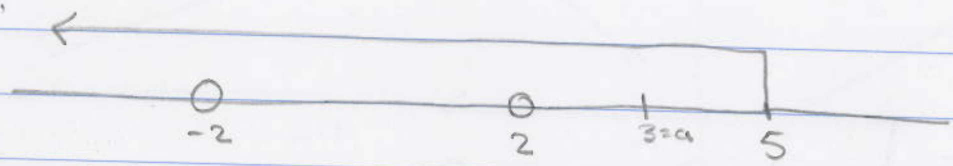
$a=3$

$\sqrt{5-t}$ is Cont on $(-\infty, 5]$

$\frac{1}{t^2-4}$ " " " $R / \{-2, 2\}$

t " " " R

e^t+3 " " " R



$I = (2, 5)$

Subject: (Date / / :التاريخ) :الموضوع

29/1/2008 Thursday

Finding The Solution

From last Class:-

$y'' + 3\sqrt{x}\, y' = x^2$

Linear

order = 2

$y \to$ Dedpendent

$x \to$ Independent

solution :- $y = \underbrace{\sim\sim}_{\text{interms of } x}$.

and it's unique on the interval $[0, \infty) \rightsquigarrow$ value of x

How to find the Solution:-

We will Use :- Laplace - Transformation :-

Def:- Let $f(x)$ be a function, the $\underbrace{\mathcal{L}\{f(x)\}}_{\text{laplace transformation}} = \int_0^\infty f(x)\, e^{-\overset{s \geq 0}{s}x}\, dx$

Ex:-

$f(x) = 1$. Find $\mathcal{L}\{1\}$

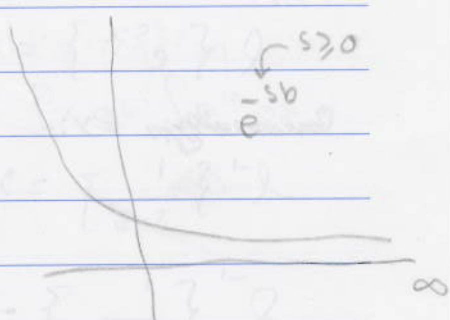
$\mathcal{L}\{1\} = \int_0^\infty 1\, e^{-sx}\, dx$

$= \lim_{b \to \infty} \frac{-1}{s} e^{-sx} \Big|_{x=0}^{x=b} = \lim_{b \to \infty} \left(\underbrace{\frac{-1}{s} e^{-sb}}_{\text{goes to zero}} + \frac{1}{s} \right)$

$= 1/s$

Note :-
the "s" is treated just like a constant.

Note:-
$\int e^{-3x} = \frac{-1}{3} e^{-3x}$



Generally :-

The first $\mathcal{L}\{f(x)\}$:.

① $\mathcal{L}\{\underbrace{C}_{\text{Constant}}\} = \frac{C}{S}$

Ex:-

Give me a function $f(x)$ such that $\mathcal{L}\{f(x)\} = 3/S$. In other words:. $\mathcal{L}^{-1}\{3/S\}$ ⇝ which function has laplace transformation.

$f(x) = 3$

Ex:- $f(x) = e^{3x}$. Find $\mathcal{L}\{f(x)\}$

$$\mathcal{L}\{e^{3x}\} = \int_0^\infty e^{3x} \cdot e^{-Sx}\, dx = \int_0^\infty e^{\overbrace{(3-S)}^{\text{Constant}}x}\, dx =$$

$$\lim_{b\to\infty} \frac{1}{3-S} e^{(3-S)x}\Big|_{x=0}^{x=b} = \lim_{b\to\infty} \left(\frac{1}{3-S} e^{(3-S)b} - \frac{1}{3-S}\right)$$

Note :-

Here we must Pick $S > 3$ in order to $3-S < 0$ ⇒ so the limit exist.

$$= \frac{-1}{3-S} = \frac{1}{S-3} \quad \text{" as long as } S > 3\text{"}$$

Ex:.

$\mathcal{L}\{e^{5x}\} = \frac{1}{S-5}$, $S<5$

~~Generally~~ Ex:-

$\mathcal{L}^{-1}\{\frac{1}{S-5}\} = ?$

$\mathcal{L}^{-1}\{\frac{1}{S-5}\} = e^{5x}$

Generally :- any real #

(2) $\mathcal{L}\{e^{ax}\} = \dfrac{1}{s-a}$

Ex:-
$\mathcal{L}^{-1}\left\{\dfrac{1}{s+10}\right\} = ?$

$f(x) = e^{-10x}$

Result:-
$\mathcal{L}\{c_1 f_1(x) \pm c_2 f_2(x)\} = c_1 \mathcal{L}\{f_1(x)\} \pm c_2 \mathcal{L}\{f_2(x)\}$

Assuming both Laplace Transformation exisits.

Proof :-
$$\mathcal{L}\{c_1 f_1(x) \pm c_2 f_2(x)\} = \int_0^\infty [c_1 f_1(x) \pm c_2 f_2(x)] e^{-sx}\, dx =$$

$$\int_0^\infty c_1 f_1(x) e^{-sx}\, dx \pm \int_0^\infty c_2 f_2(x) e^{-sx}\, dx = c_1 \int_0^\infty f_1(x) e^{-sx}\, dx \pm c_2 \int_0^\infty f_2(x) e^{-sx}\, dx$$

$$= c_1 \mathcal{L}\{f_1(x)\} \pm c_2 \mathcal{L}\{f_2(x)\}$$

$\mathcal{L}\{10 e^{-5x} + 10\} = ??$

$= 10\, \mathcal{L}\{e^{-5x}\} + 10\, \mathcal{L}\{1\}$

$= 10 \dfrac{1}{s+5} + \dfrac{10}{s}$

Laplace Trans Result is Not valid for multiplication since it depends on integration and $\int x^2 \cdot x^{10}\, dx \neq \int x^2 dx + \int x^{10} dx$

Subject: (Date / / التاريخ:)

Notation :-

because the answer is always in terms of S

✓ $\mathcal{L}\{f(x)\} = F(S)$

✓ $\mathcal{L}^{-1}\{F(s)\} = f(x)$

✓ $\mathcal{L}\{y(x)\} = Y(S)$

Result :-

$$\mathcal{L}^{-1}\{C_1 F_1(s) \pm C_2 F_2(s)\} = C_1 f_1(x) \pm C_2 f_2(x)$$

Ex:-

Find:- $\mathcal{L}^{-1}\left\{\frac{13}{s-2} - \frac{5}{s}\right\}$

$= 13\mathcal{L}^{-1}\{1/s-2\} - 5\mathcal{L}^{-1}\{1/s\}$

$= 13e^{2x} - 5$

What we did ?

$\mathcal{L}\{13e^{2x} - 5\} = 13/s-2 - 5/s$

Ex:-

$\mathcal{L}\{x\} = ?$

$= \int_0^\infty x e^{-sx}\,dx$

Integration by parts or Use the table.

The Table method:

* This method is same as intg by parts; just make calculation easier.

Derive	Integrate
dx/dy	$\int$
* one to derive untill it becomes Zero	* One easy to intgrate.

Derive		Integrate
x	$+$	e^{-sx}
1	$-$	$-\frac{1}{s} e^{-sx}$
0	$+$	$\frac{1}{s^2} e^{-sx}$

$s \geq 0$

$$\int x e^{-sx}\, dx = \lim_{b \to \infty} \left\{ \frac{-1}{s} x e^{-sx} - \frac{1}{s^2} e^{-sx} \right|_{x=0}^{x=b}$$

$= 1/s^2$

Sunday 1/2/2009

Laplace Transformation

$\mathcal{L}\{x\} = 1/s^2$

Note:- $n! = n.(n-1)(n-2) \cdots, 2.1$

$\mathcal{L}\{x^2\} = 2!/s^3 = 2/s^3$

which function has Laplace Trans to $3/s-2 + 24/s^4$

OR:-

$\mathcal{L}^{-1}\{3/s-2 + 24/s^4\}$

$= 3\mathcal{L}^{-1}\{1/s-2\} + \mathcal{L}^{-1}\{\frac{4.6}{s^4}\}$

$= 3\mathcal{L}^{-1}\{1/s-2\} + 4\mathcal{L}^{-1}\{6/s^4\} = 3e^{2x} + 4x^3$

(7) $\mathcal{L}\{f^{(n)}(x)\} = s^n F(s) - s^{n-1}f(0) - \underbrace{\cdots}_{s^{n-2}f'(0)} - f^{(n-1)}(0)$

$\mathcal{L}\{f(x)\} = F(s)$

Note:

$f^{(0)}(x) = f(x)$

at $n=1$

$\mathcal{L}\{f'(x)\} = sF(s) - f(0)$

at $n=2$

$\mathcal{L}\{f^{(2)}(x)\} = s^2F(s) - sf(0) - f'(0)$

at $n=3$

$\mathcal{L}\{f^{(3)}(x)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

Proof:-

For $n=1$:-

Fact (from math "calculus") if:-

$$\int_0^\infty K(x)\,dx \text{ exists then, } \lim_{x\to\infty} K(x) = 0$$

$$\int D'(x)\,dx = D(x) + C$$

at $n=1$. Show that $\mathcal{L}\{f'(x)\} = sF(s) - f(0)$

Trick:-

$$\int_0^\infty \left[f(x)e^{-sx}\right]' dx =$$

Note:- $\mathcal{L}\{f(x)\} = \int_0^\infty f(x)e^{-sx}\,dx = F(S)$

$$\lim_{b\to\infty} f(x)e^{-sx}\Big|_{x=0}^{x=b} = \lim_{b\to\infty} f(b)e^{-sb} - f(0) = -f(0).$$

but ~~the~~ by simplyfying the derivative inside the integration:-

$$\int_0^\infty \left[f(x)e^{-sx}\right]' dx = \int_0^\infty f'(x)e^{-sx} + -sf(x)e^{-sx}\,dx$$

$$\int_0^\infty f'(x)e^{-sx}\,dx + -s\int_0^\infty f(x)e^{-sx} = -f(0)$$

$$\mathcal{L}\{f'(x)\} - sF(s) = -f(0)$$

$$\mathcal{L}\{f'(x)\} = sF(s) - f(0)$$

Subject: ______ (Date / / التاريخ:) ______ الموضوع:

(4) Shifting Formula:-

$$\mathcal{L}\{e^{3x} \cdot x^3\} = \frac{6}{(S-3)^4}$$

$$\mathcal{L}\{e^{ax} f(x)\} = F(s)\Big|_{s \to s-a} = F(s-a)$$

$$\mathcal{L}\{e^{3x} \cdot \underbrace{x^3}_{f(x)}\} =$$

$$\mathcal{L}\{x^3\} = 6/s^4 \quad F(s).$$

Ex:-

$$\mathcal{L}\{2x^4 e^{-10x}\} = 2\,\mathcal{L}\{x^4 e^{-10x}\} \rightsquigarrow \text{use } \# 4$$

$$= 2 \frac{4!}{(S+10)^5}$$

$$\mathcal{L}^{-1}\{1/(s-13)^5\} =$$

$$\boxed{1/4! \cdot x^4} \cdot e^{13x}$$

see the following Example.

Subject: (Date / / :التاريخ) :الموضوع

Ex:-

$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} =$

$\mathcal{L}^{-1}\left\{\frac{1}{4!}\cdot\frac{4!}{s^5}\right\} = \frac{1}{4!}\cdot\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = 1/4!\cdot x^4$

Ex:- $\mathcal{L}^{-1}\left\{1/s^{10} + \frac{6}{(s+10)^4}\right\}$

$= \mathcal{L}^{-1}\left\{1/s^{10}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{(s+10)^4}\right\}$

$= \frac{1}{9!}\mathcal{L}^{-1}\left\{\frac{9!}{s^{10}}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{(s+10)^4}\right\}$

$= \frac{1}{9!}\cdot x^9 + x^3 e^{-10x}$

#2]] $\mathcal{L}\{\sin(\underbrace{k}_{\text{Konstant}} x)\} = k/(s^2+k^2)$

Ex:- $\mathcal{L}\{\sin(\sqrt{2}\cdot x)\} = \sqrt{2}/s^2+2$

#2.5]] $\mathcal{L}\{\cos(kx)\} = s/(s^2+k^2)$

$\mathcal{L}\{\cos(3x)\} = s/s^2+9$

Ex:- $\mathcal{L}^{-1}\{1/s^2+5\}$

$\frac{1}{\sqrt{5}}\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+5}\right\}$

$= \frac{1}{\sqrt{5}}\sin(\sqrt{5}\,x)$

3/2/2009

Tuesday

(4) $\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+10}\right\}$

Solution:-

$$\mathcal{L}^{-1}\frac{1}{s^2+10} = \frac{1}{\sqrt{10}}\,\mathcal{L}\,\frac{\sqrt{10}}{s^2+10} = \frac{1}{\sqrt{10}}\sin(\sqrt{10}\,x)$$

$$\to \frac{1}{\sqrt{10}}\sin(\sqrt{10}\,x)\,e^{-3x}$$

---

(5) $\mathcal{L}^{-1}\left\{\frac{s}{(s+15)^{13}}\right\}$

(1) eleminate s :-

$$\mathcal{L}^{-1}\left\{\frac{s+15-15}{(s+15)^{13}}\right\} = \mathcal{L}^{-1}\left\{\frac{s+15}{(s+15)^{13}} - \frac{15}{(s+15)^{13}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+15)^{12}}\right\} - 15\,\mathcal{L}^{-1}\left\{\frac{1}{(s+15)^{13}}\right\}$$

$$= \frac{1}{11!}x^{11}e^{-15x} - \frac{15}{12!}x^{12}e^{-15x}$$

By Using # (7):-

$$\mathcal{L}\{y^{(2)}\} = s^2Y(s) - s\,y(0) - y'(0)$$

where $\mathcal{L}\{y\} = Y(s)$

Def:- $a_n(x)\,y^{(n)} + a_{n-1}\,y^{(n-1)} + \cdots + a_1y' + a_0y = K(x)$

If $K(x) = 0$ then the given D.E is called a homogenous D.E.

$\mathcal{L}\{0\}$ = Zero.

Subject: ________ (Date / / :التاريخ) ________ :الموضوع

Ex:- (HW):-

① $\mathcal{L}^{-1}\left\{\frac{3s}{s^2+16}\right\}$

② $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}+\frac{10}{(s-2)^5}\right\}$

③ $\mathcal{L}\left\{(x+3)^2 e^{-5x}\right\}$

④ $\mathcal{L}^{-1}\left\{\cancel{1}/((s+3)^2+10)\right\}$

⑤ $\mathcal{L}^{-1}\left\{\frac{s}{(s+15)^{13}}\right\}$

⑥ $\mathcal{L}^{-1}\left\{3/2s-5\right\}$

① $3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = 3\cos(4x)$

② $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}+10\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^5}\right\} = \frac{1}{3}\sin(3x)+\frac{10}{4!}x^4e^{2x}$

x③ $\frac{1}{(x+3+5)^3}\cdot 2! = \frac{2}{(x+8)^3}$

④ $\frac{\sin(\sqrt{10}(x+3))}{\sqrt{10}}$

⑤

⑥ $3\mathcal{L}^{-1}\left\{1/2s-5\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{5}{2}}\right\} = \frac{3}{2}e^{\frac{5}{2}x}$

Subject: (Date / / التاريخ:) الموضوع:

$$y(x) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$y(x) = e^{3x} + e^{2x} \quad \forall\, x \in (-\infty, \infty)$$

General method:-

① Find $Y(s)$

② Find $\mathcal{L}^{-1}\{Y(s)\}$

Ex:-

$$\frac{3s+2}{(s+2)^2(s+4)} = \frac{a_1}{s+2} + \frac{a_2}{(s+2)^2} + \frac{a_3}{s+4}$$

$a_2 = -2$

$a_3 = -2.5 = -5/2$

a_1: pick $s=0$

$$\frac{2}{16} = \frac{a_1}{2} + \frac{-2}{4} - \frac{2.5}{4}$$

Find a_1

$a_1 =$

$$\frac{5s}{(s^2+3)(s+2)} = \frac{a_1 s + a_2}{s^2+3} + \frac{a_3}{s+2}$$

$$5s = (a_1 s + a_2)(s+2) + a_3(s^2+3)$$

Solve the following IVP :- "initial value Problem"

$y'' - 5y' + 6y = 0$. Such that $y(0) = 2$, $y'(0) = 5$

Solution :- apply Laplace to the equation.

$\mathcal{L}\{y''\} - 5\,\mathcal{L}\{y'\} + 6\,\mathcal{L}\{y\} = 0$

$s^2 Y(s) - s\,y(0) - y'(0) - 5[s\,Y(s) - y(0)] + 6\,Y(s) = 0$

solve for $Y(s)$

$s^2 Y(s) - 2s - 5 - 5s\,Y(s) + 10 + 6\,Y(s) = 0$

$Y(s)\,[s^2 - 5s + 6] = 2s - 5$

$$Y(s) = \frac{2s-5}{s^2 - 5s + 6}$$

Review on partial Fractions:-

$$\frac{2s-5}{(s-3)(s-2)} = \frac{a_1}{s-3} + \frac{a_2}{s-2}$$

$s = 3$

$a_1 = a_2 = 1$

$$\frac{2s-5}{(s-3)(s-2)} = \frac{1}{s-3} + \frac{1}{s-2}$$

Continue of Exr

$$Y(s) = \frac{1}{s-3} + \frac{1}{s-2}$$

Ex:-

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\}$$

Should observe that we should use # 8 (Derivative)

$$\frac{+1}{8}\mathcal{L}^{-1}\left\{\frac{+8s}{(s^2+16)^2}\right\} = \frac{+1}{8}\sin(4x)$$

Explanation:-

$$\mathcal{L}\{\sin(4x)\} = \frac{4}{s^2+16}$$

$$\mathcal{L}\{x\sin(4x)\} = -\left(\frac{-8s}{(s^2+16)^2}\right) = \frac{8s}{(s^2+16)^2}$$

$$\mathcal{L}\left\{\frac{1}{8}x\sin(4x)\right\} = \frac{8s}{(s^2+16)^2}$$

$$\mathcal{L}\{xe^{2x}\sin(3x)\}$$

$$= \mathcal{L}\{e^{2x}[\underbrace{x\sin(3x)}_{f(x)}]\} = F(s-2)$$

$$\mathcal{L}\{\underbrace{x\sin(3x)}_{k(x)}\} = -K'(s) = \frac{6s}{(s^2+9)^2} = -k'(s)$$

$$F(s-2) = \frac{6(s-2)}{[(s-2)^2+9]^2}$$

Solution of the Quiz :-

① $\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+2}\right) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = \sin x \, e^{-x}$

complete the square

② $\mathcal{L}^{-1}\left\{\frac{s}{(s-3)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{s-3+3}{(s-3)^4}\right\} = \mathcal{L}^{-1}\frac{1}{(s-3)^3} + 3\,\mathcal{L}^{-1}\frac{1}{(s-3)^4}$

$= \frac{1}{2}x^2 e^{3x} + \frac{1}{2}x^3 e^{3x}$

③ $\mathcal{L}\{\cos(2x)\, e^{-3x}\} = F(s+3)$

$\mathcal{L}\{\underbrace{\cos(2x)}_{f(x)}\} = \frac{s}{s^2+4} \quad F(s)$

$= \frac{s+3}{(s+3)^2+4}$

Subject: ______ (Date / / التاريخ) ______ الموضوع:

5/2/2009 Thursday

Ex:-

$$y^{(2)} + 6y' + 8y = e^{x}, \quad y(0)=3, \quad y'(0)=-5$$

$$\mathcal{L}\{y^{(2)}\} + 6\mathcal{L}\{y'\} + 8\mathcal{L}\{y\} = \mathcal{L}\{e^{x}\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 6(sY(s) - y(0)) + 8Y(s) = \frac{1}{s-1}$$

$$s^2 Y(s) - 3s + 5 + 6sY(s) - 18 + 8Y(s) = \frac{1}{s-1}$$

$$Y(s)\left[s^2 + 6s + 8\right] = \frac{1}{s-1} + 3s + 13$$

$$Y(s) = \frac{1}{(s^2+6s+8)(s-1)} + \frac{3s+13}{s^2+6s+8}$$

$$\cancel{Y(s)} \quad \frac{1}{(s+2)(s+4)(s-1)} = \frac{a_1}{s-1} + \frac{a_2}{s+4} + \frac{a_3}{s+2}$$

$$a_1 = 1/15 \qquad a_2 = 1/10 \qquad a_3 = \frac{-1}{6}$$

Subject: (Date / / التاريخ) الموضوع:

8/2/2009 Sunday

Formula Sheet

Suggested Problems:-

278 : 11, 15, 19, 21, 25, 27, 29

289 : 1, 3, 5, 7, 9, 11, 13

8:-

$$\mathcal{L}\{x f(x)\} = -F'(s) \Rightarrow \mathcal{L}\{x^n f(x)\} = (-1)^n F^{(n)}(s)$$

$$\mathcal{L}\{f(x)\} = F(s)$$

$n \geq 1$ "positive integer"

Ex:-

$\mathcal{L}\{x \sin(3x)\}$:

$$\mathcal{L}\{\sin(3x)\} = \frac{3}{s^2+9} = F(s)$$

$$\mathcal{L}\{\sin(3x)\} = -\left(\frac{-6s}{(s^2+9)^2}\right) = \frac{6s}{(s^2+9)^2}$$

we get this one from derivatives rule.

Ex:-

$$\mathcal{L}\{x^2 e^{3x}\} \overset{\#15,4}{=} \frac{2}{(s-3)^3}$$

but by using # 8:-

$$\overset{\#8}{=} \mathcal{L}\{e^{3x}\} = \frac{1}{s-3} = F(s)$$

$$\mathcal{L}\{x^2 e^{3x}\} = (-1)^2 F^{(2)}(s) = F^{(2)}(s)$$

$$F(s) = \frac{1}{s-3}$$

$$F'(s) = \frac{-1}{(s-3)^2}$$

$$F''(s) = \frac{+2(s-3)}{(s-3)^4} = \frac{2}{(s-3)^3}$$

Ex:-

$$f(x) = \begin{cases} 3 & \text{if } 0 \le x < 4 \\ x & \text{if } x \ge 4 \end{cases}$$

$f(2) = 3$

$f(10.3) = 10.3$

$$\mathcal{L}\{f(x)\} = \int_0^\infty f(x) e^{-sx} dx = \int_0^4 3 e^{-sx} dx + \int_4^\infty x e^{-sx} dx$$

For Piece Wise function We can't Use any of Laplace Trans

So, we have to use Unit Step Function.

Def:-

$a \ge 0$ "Positive a"

$$U(x-a) = \begin{cases} 0 & \text{if } 0 \le x < a \\ 1 & \text{if } x \ge a \end{cases}$$

$$U(x-3) = \begin{cases} 0 & \text{if } 0 \le x < 3 \\ 1 & \text{"} \quad x \ge 3 \end{cases}$$

$U(x) = 1 \quad , \quad U(x-\infty) = 0$

Ex:-

$$f(x) = \begin{cases} 3 & \text{if } 0 \le x \le 10 \\ \sin(x) & 10 \le x < 15 \\ x & x \ge 15 \end{cases}$$

write $f(x)$ in terms of unit step function.

Solution:-

$$f(x) = 3[U[x] - U(x-10)] + \sin x\,[U(x-10) - U(x-15)] + x[U(x-15)]$$

$$f(4) = 3[1-0] + \sin(4)(0-0) + 4[0] = 3$$

@ $x=11 \Rightarrow f(11) = \sin(11)$

$$f(11) = 3[1-1] + \sin(11)(1-0) + 11[0] = \sin(11)$$

10/2/2008 Tuesday

Page:- 279 (7.3):-

37, 39, 41, 43, 45, 47

ex:- 1

$$f(x) = \begin{cases} 2 & \text{if } 0 \le x < 10 \\ 15 & \text{if } 10 \le x < 15 \\ x & \text{if } x \ge 15 \end{cases}$$

Find $\mathcal{L}\{f(x)\}$:-

Solution:-

write $f(x)$ in terms of unit step function

$$f(x) = 2[U(x) - U(x-10)] + 15[U(x-10) - U(x-15)] + x[U(x-15)]$$

$$u(x-a) = \begin{cases} 0 & \text{if } 0 \leq x < a \\ 1 & \text{if } x \geq a \end{cases}$$

$$f(x-13)\Big|_{x=20} = 1$$

#16.011 On Formula Sheet:-

$$\mathcal{L}\{u(x-a)\} = \frac{e^{-as}}{s}$$

Ex:-

$$\mathcal{L}\{u(x-13)\} = \frac{e^{-13s}}{s}$$

from U.Fun — from Unit Function.

Prove: $$\mathcal{L}\{u(x-13)\} = \int_0^a \underbrace{0\, e^{-sx}}_{=0} dx + \int_a^\infty 1\, e^{-sx} dx = 0 - \frac{1}{s} e^{-sx}\Big|_{x=a}^{x\to\infty}$$

$$= 0 - \left(\frac{-1}{s} e^{-as}\right) = \frac{e^{-as}}{s}$$

Ex:-

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} = u(x-3)$$

By going back to ex (1):-

$$f(x) = 2 - 2u(x-10) + 15u(x-10) - 15u(x-15) + x\,u(x-15)$$

$$= 2 + 13u(x-10) - 15u(x-15) + x\,u(x-15)$$

Subject: (Date / / التاريخ:) الموضوع:

$\cancel{\mathcal{L}\{u(x-a)\,f(x)\} = e^{-as}F(s+a)}$ $\quad \mathcal{L}\{u(x-a)f(x)\} = e^{-as}\cdot\mathcal{L}\{f(x+a)\}$

by going back to our previous example "ex ①"

$$\mathcal{L}\{*\} = \mathcal{L}\{2\} + 13\,\mathcal{L}\{u(x-10)\} - 15\,\mathcal{L}\{u(x-15)\} + \mathcal{L}\{\underbrace{x}_{m(x)}\,u(x+15)\}$$

$$= 2/s + \frac{13e^{-10s}}{s} - \frac{15e^{-15s}}{s} + e^{-15s}\cdot M(s+15)$$

$$\mathcal{L}\{m(x)\} = \mathcal{L}\{x\} = \frac{1}{s^2} \Rightarrow M(s)$$

$$= \frac{2}{s} + \frac{13e^{-10s}}{s} - \frac{15e^{-15s}}{s} + e^{-15s}\cdot\frac{1}{(s+15)^2}$$

$$\mathcal{L}\{u(x-a)\underbrace{f(x-a)}_{d(x)}\} = e^{-as}D(s+a) = e^{-as}F(s)$$

$\mathcal{L}\{f(x-a)\} = \mathcal{L}\{d(x)\} = D(s) = F(s-a)$

$$D(s+a) = F(s)$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(x-a)\,f(x-a)$$

Ex:- $\mathcal{L}^{-1}\left\{\dfrac{e^{-5s}}{s^2+1}\right\}$

$$= \mathcal{L}^{-1}\left\{e^{-5s}\cdot\frac{1}{s^2+1}\right\} = U(x-5)\,\sin(x-5)$$

$$\mathcal{L}^{-1}\Big\{\overset{F(s)}{\frac{1}{s^2+1}}\Big\} = \underbrace{\sin(x)}_{f(x)} \qquad f(x-5) = \sin(x-5)$$

Ex:
$$\mathcal{L}^{-1}\left\{\frac{e^{-13s}}{s+15}\right\}$$

$$= \mathcal{L}^{-1}\left\{e^{-13s}\cdot\frac{1}{s+15}\right\} = U(x-13)\,f(x-13) = U(x-13)\,e^{-15(x-13)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+15}\right\} = e^{-15x} \quad f(x) \to f(x-13) = e^{-15(x-13)}$$

$$y^{(2)} + 5y' + 6y = U(x-2), \quad y(0)=0, \quad y'(0)=1$$

$$\mathcal{L}\{y^{(2)}\} + 5\,\mathcal{L}\{y'\} + 6\,\mathcal{L}\{y\} = \mathcal{L}\{U(x-2)\}$$

$$s^2Y(s) - s\,y(0) - y'(0) + 5\left[s\,Y(s) - y(0)\right] + 6\,Y(s) = \frac{e^{-2s}}{s}$$

$$s^2Y(s) - 1 + 5s\,Y(s) + 6\,Y(s) = \frac{e^{-2s}}{s}$$

Solve for $Y(s)$:-

$$Y(s)\left[s^2+5s+6\right] = \frac{e^{-2s}}{s} + 1$$

$$Y(s) = \frac{e^{-2s}}{s(s^2+5s+6)} + \frac{1}{s^2+5s+6}$$

Subject: (Date / / التاريخ:) الموضوع:

$$y(x) = \mathcal{L}^{-1}\left\{e^{-2s}\underbrace{\frac{1}{s(s^2+5s+6)}}_{M(s)}\right\} + \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2+5s+6}}_{W(s)}\right\}$$

$$= u(x-2)\, m(x-2) + w(x)$$

$$\frac{1}{s(s+2)(s+3)} = \frac{a_1}{s} + \frac{a_2}{s+2} + \frac{a_3}{s+3}$$

$a_1 = 1/6$

$a_2 = -\frac{1}{2}$

$a_3 = 1/3$

$$m(x) = \frac{1}{6} + \frac{-1}{2}e^{-2x} + \frac{1}{3}e^{-3x}$$

$$m(x-2) = \frac{1}{6} - \frac{1}{2}e^{-2(x-2)} + \frac{1}{3}e^{-3(x-2)}$$

$w(x) =$

$$W(s) = \frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)} \Rightarrow \frac{a_1}{s+2} + \frac{a_2}{s+3} \Rightarrow a_1 = 1, \quad a_2 = -1$$

$$w(x) = \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+3}\right\} = e^{-2x} - e^{-3x}$$

$\Rightarrow$ Solution $= u(x-2)\, m(x-2) + w(x)$

$$= u(x-2)\left(1/6 - 1/2\, e^{-2(x-2)} + 1/3\, e^{-3(x-2)}\right) + e^{-2x} - e^{-3x}$$

Subject: (Date / / التاريخ:) الموضوع:

Practice:-

$$y^{(2)} - 8y' + 12y = u(x-3)e^{5x}, \quad y(0) = y'(0) = 0$$

$$s^2 Y(s) + \cancel{sy(0)} - \cancel{y'(0)} - 8(sY(s) - \cancel{y(0)}) + 12Y(s) = e^{-3x}\,\mathcal{L}\{e^{5(x+3)}\}$$

$$Y(s)(s^2 - 8s + 12) = e^{-3x+15}\,\mathcal{L}\{e^{5x}\}$$

$$Y(s) = e^{3x+15} \cdot \frac{1}{s-5} \cdot \frac{1}{s^2 - 8 + 12}$$

$$= e^{3x+15} \cdot \frac{1}{s-5} \cdot \frac{1}{(s+2)(s+6)}$$

$$\frac{1}{(s-5)(s+2)(s+6)} = \frac{a_1}{s-5} + \frac{a_2}{s+2} + \frac{a_3}{s+6} \qquad 7 \times 11$$

$-11 \quad -7 \quad -4 \quad 4$

$$a_1 = \frac{1}{77} \qquad a_2 = \frac{-1}{28} \qquad a_3 = \frac{+1}{44}$$

$$y(x) = \mathcal{L}^{-1}\left\{ e^{3x+15}\left(\frac{1}{77}\cdot\frac{1}{s-5} + \frac{1}{44}\,\frac{1}{s+6} - \frac{1}{28}\,\frac{1}{s+2} \right) \right\}$$

$$= e^{15}\,u(x-3)\left(\frac{e^{5(x-3)}}{77} + \frac{e^{-6x(x-3)}}{44} + \frac{e^{-2(x-3)}}{-28} \right)$$

Subject: (Date / / التاريخ:) الموضوع:

12/2/2009 Thursday

Solution of the Quiz:-

(1) $$\mathcal{L}^{-1}\left\{e^{-2s}\cdot\underbrace{\frac{1}{(s+1)^2+1}}_{F(s)}\right\} = u(x-2)\cdot f(x-2)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = \underbrace{e^{-x}\sin(x)}_{f(x)}$$

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{(s+1)^2+1}\right\} = u(x-2)\,e^{-(x-2)}\sin(x-2)$$

---

(2) Similar to notes:-

(3) $y^{(2)} - 5y' + 6y = u(x-3)$

$$s^2Y(s) - sy(0) - y'(0) + (-5sY(s)) + 6Y(s) = \frac{e^{-3s}}{s}$$

$$Y(s)(s^2-5s+6) = \frac{e^{-3s}}{s}$$

$$Y(s) = e^{-3s}\cdot\frac{1}{s(s^2-5s+6)}$$

$$y(x) = \mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{s(s-3)(s-2)}\right\}$$

$$= u(x-3)\,f(x-3)$$

$$\frac{1}{s(s-3)(s-2)} = \frac{a_1}{s} + \frac{a_2}{s-3} + \frac{a_3}{s-2}$$

$$a_1 = \frac{1}{6} \quad a_2 = \frac{-1}{2} \quad a_3 = 1/3$$

Subject: (Date / / :التاريخ) :الموضوع

$$f(x) = \frac{1}{6} - \frac{1}{2}e^{2x} + \frac{1}{3}e^{3x}$$

$$f(x-3) = \frac{1}{6} - \frac{1}{2}e^{2(x-3)} + \frac{1}{3}e^{3(x-3)}$$

$$y(x) = u(x-3)\underbrace{\left[\tfrac{1}{6} - \tfrac{1}{2}e^{2(x-3)} + \tfrac{1}{3}e^{3(x-3)}\right]}_{f(x-3).}$$

Ex: #5:

$$\mathcal{L}\{u(x-3)\underbrace{e^{-4x}}_{f(x)}\} = e^{-3s}\,\mathcal{L}\{f(x+3)\} = e^{-3s}\,\mathcal{L}\{e^{-4(x+3)}\}$$

$$e^{-4x-12} = e^{-4x}\cdot e^{-12}$$

$$= e^{-3s}\,\mathcal{L}\{e^{-12}\cdot e^{-4x}\}$$

$$= e^{-12}\cdot e^{-3s}\,\mathcal{L}\{e^{-4x}\} = e^{-12}\cdot e^{-3s}\cdot\frac{1}{s+4}$$

$$= e^{-3s-12}\cdot\frac{1}{s+4}$$

$$\mathcal{L}\{\cos(3+x)\}$$

$$\mathcal{L}\{\cos(x)\cos(3) - \sin(x)\sin(3)\}$$

Subject: (Date / / التاريخ:) الموضوع:

H.W:-

① $\mathcal{L}\{u(x-2)\sin(x)\}$

② $\mathcal{L}\{u(x-3)\,e^{x}\cos(x)\}$

① $e^{-2s}\,\mathcal{L}\{\sin(x+2)\} =$

$\mathcal{L}\{\sin(x+2)\} = \mathcal{L}\{\sin(x)\cos(2) + \sin(2)\cos(x)\}$

$$= \cos(2)\cdot\frac{1}{s^2+1} + \sin(2)\,\frac{s}{s^2+1}$$

$\Rightarrow$ Solution $= e^{-2x}\left(\frac{\cos(2)}{s^2+1} + \frac{s\,(\sin(2))}{s^2+1}\right)$

② $e^{-3x}\,\mathcal{L}\{f(x+3)\}$

$f(x) = \cos(x) \Rightarrow f(x+3) = \cos(x+3) \Rightarrow \cos(x)\cos(3) - \sin(3)\sin(x)$

$\mathcal{L}\{f(x+3)\} = \frac{(\cos(3)).s}{s^2+1} - \frac{\sin(3)}{s^2+1}$

By shifting:-

$$\frac{\cos(3)\ (s-1)}{(s-1)^2+1} - \frac{\sin(3)}{(s-1)^2+1}$$

$\Rightarrow$ Solution $= e^{-3x}\left(\frac{\cos(3)\ (s-1)}{(s-1)^2+1} - \frac{\sin(3)}{(s-1)^2+1}\right)$

Subject: (Date / / :التاريخ) :الموضوع

15/2/2009 Sunday

Convolution :-

Def :- $(f_1 * f_2)(x) = \int_0^x f_1(\overset{r}{\pi})\, f_2(x-\overset{r}{\pi})\, d\pi$

Note:-

$\mathcal{L}^{-1}\{F(s) \pm K(s)\} = \mathcal{L}^{-1}\{F(s)\} \pm \mathcal{L}^{-1}\{K(s)\}$

$\mathcal{L}^{-1}\{F(s)\, K(s)\} \neq \mathcal{L}^{-1}\{F(s)\} \cdot \mathcal{L}^{-1}\{K(s)\}$

Has to do with convolution.

Ex:-

$\sin(x) * e^x = \int_0^x \sin(\pi)\, e^{x-r}$

$f_2 * f_1 = \int_0^x f_2(r) f_1(x-r)\, dr$

Result:- $f_1 * f_2 = f_2 * f_1$

Proof :-

$\int_0^x f_1(r) f_2(x-r)\, dr \Rightarrow \int_0^x f_1(r) f_2(x-r)\, dr = f_1 * f_2$

$u = x-r \rightarrow r = x-u \,,\quad \frac{du}{dr} = -1 \Rightarrow du = -dr$

To change the limits of Integration :-

@ $x=0 \Rightarrow u = x$

$r = x \rightarrow u = 0$

So :-

$-\int_x^0 f_1(x-u)\, f_2(u)\, du = \int_0^x f_2(u)\, f_1(x-u)\, du = f_2 * f_1$

Ex:- $3 * \sin(x)$

Note:

$\sin(2x) = 2\sin($

$\frac{1}{2}\sin(2x) = \sin(x$

$$3 * \sin(x) = \int_0^x 3\sin(x-r)\,dr$$

or

$$\sin(x) * 3 = \int_0^x \sin(r) \cdot (3)$$

Ex:- $10 e^x$

$$10 * e^x = \int_0^x 10\, e^r \,dr$$

$60 * \sin(x)\cos(x) =$

$$60 * \frac{1}{2}\sin(2x) = \frac{60}{2}\int_0^x \sin(2r)\,dr$$

Result:-

$$\mathcal{L}\{f_1 * f_2\} = F_1(s) \cdot F_2(s)$$

Ex:-

$$\mathcal{L}\{\sin(x) * \cos(x)\} = \mathcal{L}\left\{\int_0^x \sin(r)\cos(x-r)\,dr\right\} =$$

$$\frac{1}{s^2+1} \cdot \frac{s}{s^2+1}$$

10:-

$$\mathcal{L}\{1 * f_1(x)\} = \mathcal{L}\left\{\int_0^x f_1(r)\,dr\right\} = \frac{1}{s} \cdot F_1(s)$$

Subject: (Date / / :التاريخ) :الموضوع

Ex:- $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

Using the basics :-

① we will use partial fractions :-

$a_1 = 1$

$a_2, a_3 = ?$ @ $s=1$

$$\frac{1}{2} + (-1) = \frac{a_2}{2} + \frac{a_3}{2} \Rightarrow \frac{-1}{2} = \frac{1}{2}a_2 + \frac{1}{2}a_3 \rightarrow ①$$

and so on.

Now using #9 & # 10/10.5 :-

$$\mathcal{L}^{-1}\left\{\underset{F_1(s)}{\frac{1}{s}} \cdot \underset{F_2(s)}{\frac{1}{s^2+1}}\right\} = 1 * \sin(x) = \int_0^x \sin(r)\,dr = -\cos(r)\Big|_{r=0}^{r=x}$$

$$= -\cos(x) + \cos(0) = 1 - \cos(x)$$

Ex:- $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$

$$= \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s}}_{F_1(s)} \cdot \underbrace{\frac{1}{s(s^2+1)}}_{F_2(s)}\right\} = 1 * f_2(x) = 1 * \underbrace{(1-\cos(x))}_{\text{done previously}} =$$

$$\int_0^x [1-\cos(r)]\,dr = r - \sin(r)\Big|_0^x = x - \sin(x) - (0-0) = x - \sin(x)$$

Ex:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+4)}\right\} =$$

by integrating the previous result:

$$= \frac{1}{2}r^2 + \cos(r)\Big|_{r=0}^{r=x}$$

$$= \frac{1}{2}x^2 + \cos(x) - 1$$

Explanation:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{1/s \cdot 1/s(s^2+1)\right\} = 1 * (1-\cos(x)) =$$

$$\int_0^x 1-\cos(r)\,dr = r - \sin(r)\Big|_{r=0}^{r=x} = x - \sin(x)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s}}_{F_1(s)} \cdot \underbrace{\frac{1}{s^2(s^2+1)}}_{F_2(s)}\right\} = 1 * \mathcal{L}^{-1}\{F_2(s)\} = 1 * (x-\sin$$

$$= \int_0^x r - \sin(r)\,dr = (1/2)\cdot r^2 + \cos(r)\Big|_0^x = \frac{1}{2}x^2 + \cos(x) - 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4(s^2+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s}}_{F_1(s)} \cdot \underbrace{\frac{1}{s^3(s^2+1)}}_{F_2(s)}\right\} = 1 * \left(-1 + \frac{1}{2}x^2 + \cos(x)\right)$$

$$= \int_0^x -1 + \frac{1}{2}r^2 + \cos(r)\,dr = -r + 1/6\, r^3 + \sin(r)\Big|_0^x$$

Subject: (Date / / التاريخ:) الموضوع:

17/2/2009 Tuesday

Ex:-

$$\mathcal{L}\left\{\int_0^t t e^{2r}\, dr\right\} =$$

$$\mathcal{L}\left\{t\int_0^t e^{2r}\, dr\right\} = \mathcal{L}\left\{t \frac{1}{2} e^{2r}\Big|_{r=0}^{r=t}\right\} = \mathcal{L}\left\{t\left(\tfrac{1}{2} e^{2t} - \tfrac{1}{2}\right)\right\}$$

$$= \frac{1}{2}\mathcal{L}\{te^{2t}\} - \tfrac{1}{2}\mathcal{L}\{t\} = \frac{1}{2}\frac{1}{(S-2)^2} - \frac{1}{2}\frac{1}{S^2}$$

But by Using Convolution:-

$$\mathcal{L}\left\{t\int_0^t e^{2r}\, dr\right\} = \mathcal{L}\{t(1 * e^{2t})\}$$

$$\mathcal{L}\{\underbrace{1 * e^{2t}}_{f(t)}\} = \frac{1}{S}\cdot\frac{1}{S-2} , \boxed{\frac{1}{S^2-2S}} \rightarrow F(s)$$

$$, \mathcal{L}\{t(1 * e^{2t})\} = -F'(s)$$

$$-F'(s) = \frac{-(-2S+2)}{(S^2-2S)^2} = \frac{2S-2}{(S^2-2S)^2}$$

Example:- #10

Find $y(x)$. $y(x) = 3 + \int_0^x y(r)\, dr$

$$Y(s) = \frac{3}{S} + \frac{Y(s)}{S}$$

$$Y(s)[1 - 1/s] = 3/s$$

$$Y(s)\left(\frac{S-1}{S}\right) = \frac{3}{S}$$

$$Y(s) = \frac{3}{S-1}$$

$$y(x) = 3\,\mathcal{L}^{-1}\{1/_{S-1}\}$$

$$y(x) = 3e^x$$

Example

Find $K(w)$

$$K'(w) = e^{w} + \int_0^{w} K(t)\,dt \quad , K(0) = 0$$

$$S K(s) - \cancel{K(0)} = \frac{1}{S-1} + \frac{K(s)}{s}$$

$$K(s)\left[s - \frac{1}{s} \right] = \frac{1}{S-1}$$

$$K(S) = \frac{s}{(S-1)^2 (s+1)}$$

$$\frac{S}{(S-1)^2 (S+1)} = \frac{a_1}{S+1} + \frac{a_2}{S-1} + \frac{a_3}{(S-1)^2}$$

$a_1 = -1/4$

$a_3 = 1/2$

$$0 = -1/4 + \frac{a_2}{-1} + \frac{1/2}{1} \Rightarrow \text{solve for } a_2 \Rightarrow a_2 = 1/4$$

$$K(w) = \frac{-1}{4} e^{-w} + 1/4\, e^{w} + \frac{1}{2} w e^{w}$$

Example

$y'' + 4y = x \quad , y(0) = y'(0) = 0$

$$S^2 Y(s) - \cancel{S y(0)} - \cancel{y'(0)} + 4Y(s) = \frac{1}{S^2}$$

$$Y(s)(s^2 + 4S) = \frac{1}{S^2}$$

$$Y(s) = \frac{1}{s^2 (S^2 + 4S)}$$

$$y(x) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2 (s^2 + 4s)} \right\}$$

Subject: (Date / / :التاريخ) :موضوع

Example

Find $K(w)$

$K'(w) = e^{w} + \int_0^w K(t)\,dt \quad , K(0) = 0$

$S K(s) - \cancel{K(0)} = \frac{1}{S-1} + \frac{K(s)}{s}$

$K(s)\left[s - \frac{1}{s} \right] = \frac{1}{S-1}$

$K(S) = \frac{S}{(S-1)^2 (S+1)}$

$\frac{S}{(S-1)^2 (S+1)} = \frac{a_1}{S+1} + \frac{a_2}{S-1} + \frac{a_3}{(S-1)^2}$

$a_1 = -1/4$

$a_3 = 1/2$

$0 = -1/4 + \frac{a_2}{-1} + \frac{1/2}{1} \Rightarrow$ solve for $a_2 \Rightarrow a_2 = 1/4$

$K(w) = \frac{-1}{4} e^{-w} + 1/4 e^{w} + \frac{1}{2} w e^{w}$

Example

$y'' + 4y = x \quad , \quad y(0) = y'(0) = 0$

$S^2 Y(s) - \cancel{S y(0)} - \cancel{y'(0)} + 4Y(s) = \frac{1}{S^2}$

$Y(s)(s^2 + 4S) = \frac{1}{S^2}$

$Y(s) = \frac{1}{s^2 (s^2 + 4S)}$

$y(x) = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 (s^2 + 4s)} \right\}$

Subject: (Date / / :التاريخ) :الموضوع

$$y(x) = \left\{ \frac{1}{s} \cdot \frac{1}{s(s^2+4)} \right\}$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s(s^2+4)} \right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{2}{s(s^2+4)} \right\} = \frac{1}{2}\underbrace{\int_0^x \sin(2r)\, dr}_{f(x)}$$

$$y(x) = 1 * f(x)$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos(2r)\right]\Bigg|_{r=0}^{r=x} = -\frac{1}{4}\left(\cos(2x) - 1\right) =$$

$$\underbrace{\frac{1}{4} - \frac{1}{4}\cos(2x)}_{f(x)} = 1 * f(x)$$

Note:

$$\mathcal{L}^{-1}\left\{ \frac{1}{s} \cdot F(x) \right\} = f(x)$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2} F(x) \right\} = 1 * f(x) = K(x)$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s} \cdot \frac{1}{s(s^2+4)} \right\} = 1 * \left(\frac{1}{4} - \frac{1}{4}\cos(2x)\right)$$

$$= \int_0^x \frac{1}{4} - \frac{1}{4}\cos(2r)\, dr = \left(\frac{1}{4}r - \frac{1}{8}\sin(2r)\right)\Bigg|_{r=0}^{r=x}$$

$$= \frac{1}{4}x - \frac{1}{8}\sin(2x) - 0 = \frac{1}{4}x - \frac{1}{8}\sin(2x)$$

Subject: (Date / / التاريخ:) الموضوع:

19/2/2009 Thursday

Quiz Solution: (3)

$$\boxed{y' - y = x\, u(x-1)}$$

$$s Y(s) - Y(s) = e^{-s}\, \mathcal{L}\{x+1\}$$

$$Y(s)(s-1) = e^{-s}\left(1/s^2 + 1/s\right)$$

$$Y(s) = e^{-s}\underbrace{\left(\frac{1}{s^2(s-1)} + \frac{1}{s(s-1)}\right)}_{F(s)}$$

$$y(x) = u(x-1)\, f(x-1)$$

$$\frac{1}{s^2(s-1)} = \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s-1}$$

$a_2 = -1$ $a_3 = 1$ $a_1 = -1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = -1 - x - e^{-x}$$

by fraction

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2} \cdot \frac{1}{s-1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s-1}\right\} = \int_0^x e^r\, dr =$$

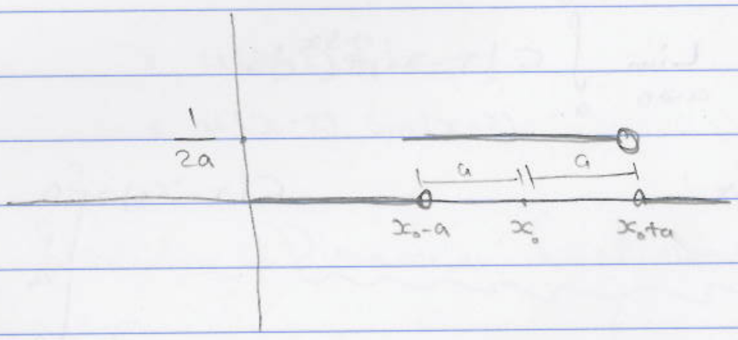
Subject: (Date / / التاريخ:) الموضوع:

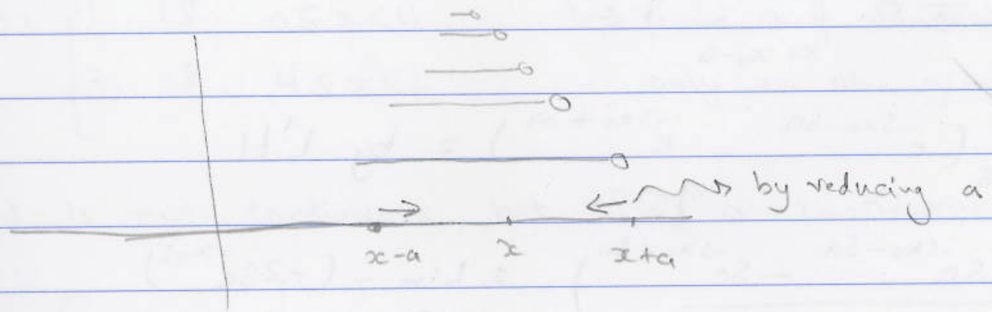
24/2/2009 Tuesday

Def:- Delta Function. "Explosion Function".

let $x_0 > a > 0$

$$\sigma(x-x_0) = \begin{cases} 0 & \text{if } 0 \le x \le x_0 - a \\ \frac{1}{2a} & \text{if } x_0 - a \le x \le x_0 + a \\ 0 & \text{if } x \geq x_0 + a \end{cases}$$



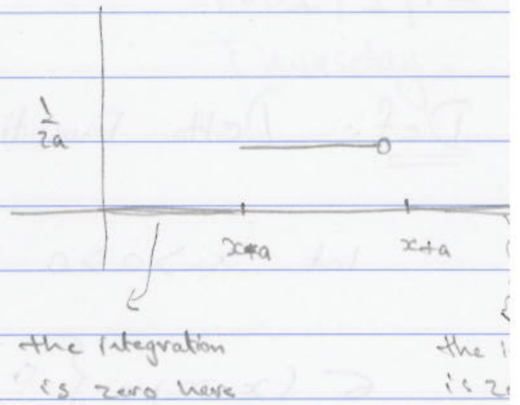


$$\sigma(x-10) = \begin{cases} 0 & \text{if } 0 \le x < 10 - a \quad \text{"For some } 0<a<10\text{"} \\ \frac{1}{2a} & \text{if } 10 - a \le x < 10 + a \\ 0 & \text{if } x \geq 10 + a \end{cases}$$

Subject: (Date / / :التاريخ) :الموضوع

Ex:- $\int_0^\infty \delta(x-x_0)\,dx$

$= \int_{x_0-a}^{x_0+a} \frac{1}{2a}\,dx = 1$



more details:-

$\int_{x_0-a}^{x+a} \frac{1}{2a} = \frac{1}{2a}(x+a-x+a)$

$= 1$

Note:-

$\int_a^b c\,dx = c(b-a)$

Def:- $\mathcal{L}\{\delta(x-x_0) = \lim_{a\to 0} \int_0^\infty \delta(x-x_0)\,e^{-sx}\,dx$

$= \lim_{a\to 0} \int_{x-a}^{x+a} \frac{1}{2a} e^{-sx}\,dx$

$\delta(x-x_0) = \begin{cases} 0 & 0\le x< \\ \frac{1}{2a} & x-a\le x \\ 0 & x> \end{cases}$

$= \frac{1}{2a}\left(\frac{-1}{s} e^{-sx} \Big|_{x=x_0-a}^{x=x_0+a} \right) =$

$\lim_{a\to 0} \frac{-1}{2as}\left(e^{-sx_0-sa} - e^{-sx_0+sa} \right) =$ by L'H

$\lim_{a\to 0} -\left(\frac{-se^{-sx_0-sa} - se^{-sx_0+sa}}{2s} \right) = \lim_{a\to 0} - \frac{(+2se^{-x_0s})}{2s}$

$= e^{-x_0 s}$

Example:-

$\mathcal{L}\{\delta(x-13)\} = e^{-13s}$

$\mathcal{L}\{\delta(x)\} = 1$

Subject: (Date / / :التاريخ) :الموضوع

Solve :-

$y'' + y = \delta(x-2)$, $y(0) = y'(0) = 0$

$s^2 Y(s) - \cancel{s y(0)} - \cancel{y'(0)} + Y(s) = e^{-2s}$

$Y(s)(s^2+1) = e^{-2s}$

$Y(s) = e^{-2s} / s^2+1$

$y(x) = \mathcal{L}^{-1}\left\{ e^{-2s} \cdot \frac{1}{s^2+1} \right\}$

$= u(x-2) f(x-2)$

$= u(x-2) \sin(x-2)$

Periodic $\underbrace{\text{Piecewise Continuous}}$ on $[0, \infty)$

$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 4 \\ 3 & \text{if } 4 \le x < 6 \end{cases}$ → means that function is continuous only on the given interval.

$f(x)$ is not continuous but $f(x)$ is continuous on $0 \le x < 4$

$$f(x) = \begin{cases} 1 & 0 \le x < 4 \\ 3 & 4 \le x < 6 \end{cases} \quad \text{1st Period}$$

$$\begin{cases} 1 & 6 \le x < 10 \\ 3 & 10 \le x < 12 \end{cases} \quad \text{2nd period}$$

to infinity

3rd period

$$\begin{cases} 1 & 12 \le x \le 16 \\ 3 & 16 \le x < 18 \end{cases}$$

Subject: (Date / / :التاريخ) :الموضوع

Solve :-

$y'' + y = \delta(x-2)$, $y(0) = y'(0) = 0$

$s^2 Y(s) - \cancel{s y(0)} - \cancel{y'(0)} + Y(s) = e^{-2s}$

$Y(s)(s^2+1) = e^{-2s}$

$Y(s) = e^{-2s} / s^2+1$

$y(x) = \mathcal{L}^{-1}\left\{ e^{-2s} \cdot \frac{1}{s^2+1} \right\}$

$= u(x-2) f(x-2)$

$= u(x-2) \sin(x-2)$

Periodic Piecewise Continuous on $[0, \infty)$

$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 4 \\ 3 & \text{if } 4 \le x < 6 \end{cases}$

means that function is continuous only on the given interval.

$f(x)$ is not continuous but $f(x)$ is continuous on $0 \le x < 4$

$f(x) = \begin{cases} 1 & 0 \le x < 4 \\ 3 & 4 \le x < 6 \end{cases}$ 1st Period

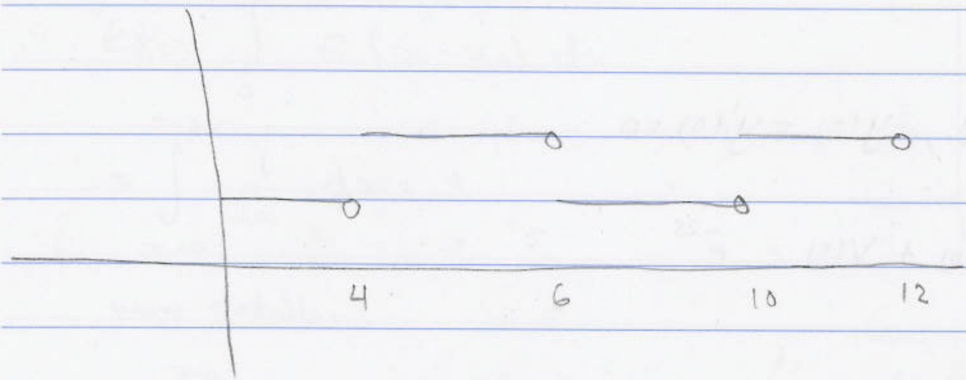
$\begin{cases} 1 & 6 \le x < 10 \\ 3 & 10 \le x < 12 \end{cases}$ 2nd period

to infinity

3rd period

$\begin{cases} 1 & 12 \le x < 16 \\ 3 & 16 \le x < 18 \end{cases}$

Subject: (Date / / :التاريخ) :موضوع



$T = \text{Period} = 6$

$$\mathcal{L}\{f(x)\} = \frac{1}{1-e^{-Ts}} \int_0^T f(x)\, e^{-sx}\, dx$$

Ex:

you have periodic function.

$$f(x) = \begin{cases} 1 & 0 \leq x < 4 \\ 3 & 4 \leq x < 6 \end{cases}$$

$$\mathcal{L}\{f(x)\} = \frac{1}{1-e^{-6s}} \int_0^6 f(x)\, e^{-sx}\, dx = \frac{1}{1-e^{-6s}} \left(\int_0^4 1\, e^{-sx}\, dx + \int_4^3 3e^{-sx}\, dx \right.$$

Subject: (Date / / :التاريخ) :الموضوع

Introduction :-

$$\left.\begin{aligned} \underline{f_1(t)}\,x + \underline{f_2(t)}\,y &= \underline{f_3(t)} \\ \underline{K_1(t)}\,x + \underline{K_2(t)}\,y &= \underline{K_3(t)} \end{aligned}\right] \text{ Solve } \quad \begin{aligned} x &= \_\_\_ \\ y &= \_\_\_ \end{aligned}$$

Note :- The underlined functions are Known.

$$tx + 3y = t^2$$

$$-2tx + ty = \sin(t)$$

$$6y + ty = 2t^2 + \sin(t)$$

$$y = \frac{2t^2 + \sin(t)}{6+t}$$

Cre C. Raman :-

$$x = \frac{\det \begin{bmatrix} t^2 & 3 \\ \sin t & t \end{bmatrix}}{\det \begin{bmatrix} t & 3 \\ -2t & t \end{bmatrix}} \qquad \frac{t^3 - 3\sin t}{t^2 - 6t}$$

$$y = \frac{\det \begin{bmatrix} t & t^2 \\ -2t & \sin t \end{bmatrix}}{\det \begin{bmatrix} t & 3 \\ -2t & t \end{bmatrix}} = \frac{t \sin t + 2t^3}{t^2 + 6t}$$

Subject: First Exam on Tuesday, March 17, at 5:00 Nab12, 13 (Date / / التاريخ) موضوع:

1/3/2009 Sunday

C. Raman.

$x(t) + y'(t) = 3t \rightarrow (1)$

$2x'(t) - y(t) = 6 \rightarrow (2), \quad x(0) = 0, \quad y(0) = 1$

* We can think of the question as :: x, y are functions in term of t.

* One way: solve for $y(t)$ in (2)

$$y(t) = 2x'(t) - 6$$

$$y'(t) = 2x''(t) \longrightarrow (3)$$

plug (3) in (1) "Or substitute $2x''(t)$ for $y'(t)$ in #1"

$2x''(t) + x(t) = 3t$

* Standard method :-

(1) Take Laplace Trans of 1.st and 2nd Equation.

$X(s) + SY(S) - y(0) = 3/s^2$

$2[SX(s) - x(0)] - Y(s) = 6/s$

$X(s) + SY(S) = 3/s^2 - 1$

$2SX(s) - Y(S) = 6/S$

Subject: (Date / / التاريخ:) الموضوع:

by C. Ramar

$$X(s) = \frac{\det \begin{array}{cc} X(s) & Y(s) \\ \begin{bmatrix} \frac{3}{s^2}+1 & s \\ 6/s & -1 \end{bmatrix} \end{array}}{\det \begin{array}{cc} X(s) & Y(s) \\ \begin{bmatrix} 1 & s \\ 2s & -1 \end{bmatrix} \end{array}}$$

$$Y(s) = \frac{\det \begin{array}{cc} Y & Y \\ \begin{bmatrix} 1 & \frac{3}{s^2}+1 \\ 2s & 6/s \end{bmatrix} \end{array}}{\det \begin{array}{cc} X & Y \\ \begin{bmatrix} 1 & s \\ 2s & -1 \end{bmatrix} \end{array}}$$

$$X(s) = \frac{\frac{-3}{s^2} - 1 - \frac{6}{\cancel{s}}}{-1-2s^2} = \frac{\frac{-3}{s^2}-7}{-1-2s^2} = \frac{-3-7s^2}{s^2(-1-2s^2)}$$

$$= \frac{7s^2+3}{s^2(2s^2+1)}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{7s^2+3}{s^2(2s^2+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{7\cancel{s^2}}{\cancel{s^2}(2s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s^2(2s^2+1)}\right\}$$

$$= \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1/2}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1/2)}\right\}$$

$$= \frac{7}{2\sqrt{1/2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{1/2}}{s^2+1/2}\right\} + 3/2\,\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1/2)}\right\}$$

$$= \frac{7}{2\sqrt{0.5}}\cdot\sin(\sqrt{0.5}\,t) + \frac{3}{2\sqrt{0.5}}\mathcal{L}^{-1}\left\{\frac{\sqrt{0.5}}{s^2(s^2+1/2)}\right\}$$

Subject: (Date / / التاريخ) موضوع:

$$\mathcal{L}^{-1}\left\{\frac{\sqrt{0.5}}{s(s^2+1/2)}\right\} = f_1(t) * f_2(t) = \frac{1}{s} * \sin(\sqrt{0.5}\,t) =$$

$$\int_0^t \sin(\sqrt{0.5}\,r)\,dr = \frac{-1}{\sqrt{0.5}}\cos(\sqrt{0.5}\,r)\Big|_{r=0}^{r=t}$$

$$\frac{-1}{\sqrt{0.5}} \cdot \cos(\sqrt{0.5}\,t) + \frac{1}{\sqrt{0.5}}$$

$$\mathcal{L}^{-1}\left\{\frac{\sqrt{0.5}}{s^2(s^2+1/2)}\right\} = \int_0^t \left[\frac{-1}{\sqrt{0.5}}\cos(\sqrt{0.5}\,r) + 1/\sqrt{0.5}\right] dr$$

$$= \frac{-\sin(\sqrt{0.5}\,r)}{0.5} + \frac{1}{\sqrt{0.5}}\,r\,\Big|_{r=0}^{r=t}$$

$$= -2\sin(\sqrt{0.5}\,t) + 1/\sqrt{0.5}\,t$$

Now:-

$$x(t) = \frac{7}{2\sqrt{0.5}}\sin(\sqrt{0.5}\,t) + \frac{3}{2\sqrt{0.5}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{0.5}}{s^2(s^2+1/2)}\right\}$$

$$= \frac{7}{2\sqrt{0.5}}\sin(\sqrt{0.5}\,t) + \frac{3}{2\sqrt{0.5}}\left[-2\sin(\sqrt{0.5}\,t) + (1/\sqrt{0.5})t\right.$$

from (2)

$$2x'(t) - y(t) = 6$$

$$y(t) = 2x'(t) - 6$$

$$x'(t) = 7/2 \cdot \cos(\sqrt{0.5}\cdot t) + 3/2\sqrt{0.5}\left[-2\sqrt{0.5}\cos(\sqrt{0.5}\,t) + \frac{1}{\sqrt{0.5}}\right]$$

$$y(t) = 2[\quad] - 6$$

Ex:-

* During all previous lecture we were dealing with $e^{f(x)}$

* but what would happen in case of 3^{5x} (ie)

* by using Log Properties:-

$$\mathcal{L}\{ e^{\ln 3^{5x}} \} = \mathcal{L}\{ e^{5x \ln 3} \}$$

$$\mathcal{L}\{ 7^{2x} \} = \mathcal{L}\{ e^{\ln 7^{2x}} \} = \mathcal{L}\{ e^{2\ln(7) x} \} = \frac{1}{s - 2\ln(7)}$$

Chapter Four :-

* Undetermined Co-efficient Method :-

Def: Independent :-

$f_1, f_2, f_3, \ldots, f_n$ are function.

let $\underbrace{c_1, c_2, \ldots, c_n}_{\text{constant.}} \in \mathbb{R}$

$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$ Linear Combination of $f_1, \ldots, f_n$.

Ex:-

$\underset{f_1(x)}{2x}, \underset{f_2(x)}{e^x}$

$\underset{c_1}{3}(\underset{f_1}{2x}) + \underset{c_2}{6}\underset{f_2}{e^x}$

Subject: (Date / / :التاريخ) :الموضوع

So Independent is:-

$f_1, f_2, \ldots f_n$ are independent if none of the f_i's is a linear combination of the other f_i's.

Dependent :-

$f_1, f_2, \ldots f_n$ are dependent if at least one of the f_i's is a linear combination of the other f_i's.

Ex

$\sin(x), \cos(x)$ Is it linear or not?

Since we can't say $\cos(x) = C . \sin(x)$

So, both of them are independent.

Ex:-

$e^{3x}, \cos(4x)$

Both are independent.

Ex:- $x, 3x^2$

$3x^2 = 1/3\, x . x$ " we can't say that since "x" is variable".

So, again it's independent.

Ex:- $\sin(x)\cos(x) = \sin(2x)$

$\sin(2x) = \underbrace{2}_{C_1} \sin(x) \cos(x) \Rightarrow$ given and known formula.

Ex: e^x, e^{-x}

Again, ~~___~~ independent.

Subject: (Date / / التاريخ:) الموضوع:

Ex:

e^{4x}, e^{-4x}

Independent.

Ex:

$e^{x}, e^{-3x}, 6e^{-3x}$

Since: $6e^{-3x} = \underset{c_1}{0}\,e^{x} + \underset{c_2}{6}\,e^{-3x}$

Result:-

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

- y = Dependent Variable.
- x = Independent "
- Homogenuis Diff Equation, and it's Linear.
- Has exactly $\underline{n}$ indpendent solutions, say $y_1(x), y_2(x), \dots, y_n(x)$. If every other solution to the Diff equation is just linear combination of $y_1(x), y_2(x), \dots, y_n(x)$

In fact, every linear combination of $y_1, y_2, \dots, y_n$ is a solution to D.E.

Example:- Find the general solution to:- $y^{(2)} - 4y = 0$

Assume $y = e^{mx}$ to be solution. Find m.

$y' = me^{mx}$

$y'' = m^2e^{mx}$

$m^2e^{mx} - 4e^{mx} = 0$

Solve for m.

$e^{mx}(m^2 - 4) = 0$

$e^{mx} \neq 0$

$\boxed{m^2 - 4 = 0}$ → Characteristic Polynomail of D.E.

$(m+2)(m-2) = 0$

$m = \pm 2$

$y_1 = e^{2x}$

$y_2 = e^{-2x}$

general Solution:-

$y = c_1e^{2x} + c_2e^{-2x}$

Subject: (Date / / :التاريخ) :الموضوع

* Undetermine Co-efficient method will be used to solve DE of this form:-

$C_n y^{(n)} + C_{n-1} y^{(n-1)} + \cdots + C_1 y' + C_0 y = K(x)$

where $C_n, C_{n-1}, \cdots C_0$ are constant, and it will be solved using

$y = e^{x}$

Ex1. Solve:-

$y^{(2)} - 6y' + 8y = 0$

$y = e^{x}$

Characteristic D.E:-

$m^2 - 6m + 8$

$m^2 - 6m + 8 = 0$

$(m - 2)(m-4) = 0$

$m = 2 \Rightarrow y_1 = e^{2x}$

$m = 4 \Rightarrow y_1 = e^{4x}$

$y_g = C_1 e^{2x} + C_2 e^{4x}$

Subject: (Date / / :التاريخ) :الموضوع

5/3/2009 Thursday

4.3 :-

Find General Solution :-

$y^{(2)} - 6y' + 8y = 0$

$y = me^{x}$

Find Charac.

$\text{Chara}(DE) = m^2 - 6m + 8$

$m^2 - 6m + 8 = 0$

$(m-2)(m-4) = 0$

$m = 2 \rightarrow y_1 = e^{2x}$

$m = 4 \rightarrow y_2 = e^{4x}$

$y_g = C_1 e^{2x} + C_2 e^{4x}, \quad C_1 \, \& \, C_2 \in \mathbb{R}$

Q: Is $y = 1.2e^{4x}$ a solution? Yes since $C_1 = 0$, $C_2 = 1.2$

Q Is $y = e^{3x}$ a solution? NO.

Ex:- $y^{(2)} - 6y' + 9y = 0$

$\text{Char}(DE) = m^2 - 6m + 9$

$m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m = 3 \rightarrow e^{3x} \rightarrow y_1$

$m = 3 \rightarrow e^{3x} \rightarrow y_2 = xe^{3x}$

$y_g = C_1 e^{3x} + C_2 x e^{3x}$

Subject: ______ (Date / / التاريخ:) ______ الموضوع:

15/3/2009 Sunday

Ex:-

$y^{(3)} - y' = 0$

Solution:-

$y = e^{mx}$. Find m.

Charac (D.E) $= m^3 - m$

Set Char (D.E) $= 0$

$m^3 - m = 0$

$m(m^2 - 1) = 0$

$m(m-1)(m+1) = 0$

$m = 0 \rightarrow e^0 = 1$

$m = 1 \rightarrow e^x$

$m = -1 \rightarrow e^{-x}$

General Solution:-

$y_g = c_1 + c_2 e^x + c_3 e^{-x}$, $c_1, c_2, c_3 \in \mathbb{R}$

Ex:- Is $10 - 3e^{-x}$ is Solution?

Ans: Yes.

Ex:- Is $y = 3^2 + 10e^x - 7e^{-x}$ a solution?

Ans:- Yes

Example:-

$y^{(3)} + 8y^{(2)} + 16y' = 0$

solution:-

$y = e^{mx}$

Char (D.E) $= m^3 + 8m^2 + 16m$

Set Char (D.E) $= 0$

$m^3 + 8m^2 + 16m = 0$

$m(m^2 + 8m + 16) = 0$

$m(m+4)(m+4) = 0$

Subject: (Date / / التاريخ:) الموضوع:

$m=0 \rightarrow e^{0}=1=y_1$

$m=-4 \rightarrow e^{-4x}=y_2$

$m=-4 \rightarrow xe^{-4x}=y_3$

note :-
in case if we have another $m=-4$, Then $y_4=x^2e^{-4x}$

$y_g=C_1+C_2e^{-4x}+C_3xe^{-4x}$

If we try to use Laplace :-

$y^{(3)}-y'=0$

$S^3Y(s)-\underbrace{S^2y(0)}_{C_1}-\underbrace{Sy'(0)}_{C_2}-\underbrace{y''(0)}_{C_3}-[SY(s)-\underbrace{y(0)}_{C_1}]=0$

$Y(s)[S^3-S]=C_1S^2+C_2S+C_3-C_1$

$Y(s)=\dfrac{C_1S^2+C_2S+C_3-C_1}{s(S^2-1)}=\dfrac{C_1S^2+C_2S+C_3-C_1}{s(s-1)(s+1)}=\dfrac{a_1}{s}+\dfrac{a_2}{s-1}+\dfrac{a_3}{s+1}$

$y(x)=\mathcal{L}^{-1}\left\{\dfrac{a_1}{s}+\dfrac{a_2}{s-1}+\dfrac{a_3}{s+1}\right\}$

$y(x)=a_1\cdot 1+a_2e^{-x}+a_3e^{x}$

Example:-

$y^{(2)}+y=0$

Solution:-

$y=e^{mx}$

Char (D.E) $=m^2+1$

set $m^2+1=0$

Note:-

Complex Number :-

$i^n=\begin{cases} i & \text{if } n \bmod (4)=1 \\ 1 & \text{if } n \bmod (4)=0 \\ -i & \text{if } n \bmod (4)=2 \\ -1 & \text{if } n \bmod (4)=3 \end{cases}$

* Any Complex number has the form $a+ib$
(a: real part, b: imaginary part.)

Fact :- $e^{xi}=\cos(x)+i\sin(x)$, $x\in\mathbb{R}$

$e^{\pi i}=-1$

Amazing Formula !

Example:-

$$\boxed{e^{a+xi} = e^{a} \cdot e^{xi} = e^{a}\left[\cos(x) + i\sin(x)\right]}$$

by going back to previous example:-

$m^2+1=0$

$m^2=-1$

$m=\pm\sqrt{-1}$

$=\pm i$

$m=a+bi \Rightarrow a=0,\ b=1$

Two independent Solution:-

* $y_1=e^{ax}(\cos(bx)) \Rightarrow y_1=\cos(bx)$, since $a=0$

$y_2=e^{ax}(\sin(bx)) \Rightarrow y_2=\sin(bx)$, since $a=0$

$y_g=C_1\cos(bx)+C_2\sin(bx)$

Example:-

$y^{(2)}+y'+4y=0$

$y=e^{mx}$

Char (D.E) $= m^2+m+4$

$m^2+m+4=0$

$m=\dfrac{-1\pm\sqrt{-15}}{2} = a + bi$ (Since imaginary, take one solution)

Real # Real # Imaginary

$=-1/2+\dfrac{\sqrt{-15}}{2}$

$=\underbrace{\dfrac{-1}{2}}_{a}+\underbrace{\dfrac{\sqrt{15}}{2}}_{b}\,i$

Subject: (Date / / التاريخ:) الموضوع:

$$y_1 = e^{ax}\cos(bx)$$
$$= e^{-1/2\,x}\cos\left(\frac{\sqrt{15}}{2}x\right)$$

$$y_2 = e^{ax}\sin(bx)$$
$$= e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{15}}{2}x\right)$$

$$y_g = C_1 e^{-1/2x}\cos(bx) + C_2 e^{-1/2x}\sin\left(\frac{\sqrt{15}}{2}x\right)$$
$$= e^{-1/2\,x}\left[C_1\cos(bx) + C_2\sin\left(\frac{\sqrt{15}}{2}x\right)\right].$$

Example:-

$y^{(2)} + 9y = 0$

$y = e^{mx}$

Char (D.E) $= m^2 + 9$

$m^2 + 9 = 0$

$m = \sqrt{-9}$ {enough one solution}

$m = \sqrt{9}\,i = 3i = \underbrace{0}_{a} + \underbrace{3}_{b}i$

$y_1 = \cos(3x)$

$y_2 = \sin(3x)$

$y_g = C_1\cos(3x) + C_2\sin(3x)$

19/3/2009 Thursday

Continue of ...

Undetermined Co-effecient Method.

$a_n y^{(n)} + \dots + a_0 y = K(x)$

If $K(x) = 0$, we are done "Homogenious"

$a_n, a_{n-1}, \dots, a_0 \rightarrow$ are constant.

We used the UCM if $K(x)$ is one of the following:-

1) Constant
(2) Polynomial $\underbrace{d_n x^n + d_{n-1} x^{n-1} + \dots + d_1 x + d_0}_{\text{all exponents are positive integers}}$ → constant.
(3) $\sin(x)$
(4) $\cos(x)$
(5) e^{mx}

Ex:- Solve:- $y^{(2)} + 5y' + 6y = 12$. Find General Solution.

Solution:-

(1) Find the associated homogeneous system:-

$y^{(2)} + 5y' + 6y = 0$

$y = e^{mx}$

Char (DE) $= m^2 + 5m + 6$

Set $m^2 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

$m = -2 \quad y_1 = e^{-2x}$

$m = -3 \quad y_2 = e^{-3x}$

Subject: (Date / / التاريخ:) الموضوع:

$y_h = C_1 e^{-2x} + C_2 e^{-3x}$ h: homogenious solution.

our general Solution :

$y_g = y_h + y_p$ P= Particular solution.

→ Stare at $K(x)$:-

→ Is $K(x)$ a solution to the homogenous part by staring?

- No

→ $y_p = a$, "a" is constant.

→ Find a :

- $y_p' = 0$
- $y_p'' = 0$

if we plug the value of y_p, y_p', y_p'' in the main equation.

$0 + 0 + 6a = 12 \Rightarrow a = 2$

$y_g = y_h + y_p$

$= C_1 e^{-2x} + C_1 e^{-3x} + 2$

Is 4 a solution? NO

Is $3e^{-2x} + 2$ a solution? Yes.

Example:-

Solve:-

$$y^{(2)} + 5y' + 6y = 3x$$

Solution:-

Find the solution to the associated homogenious system

$$y^{(2)} + 5y' + 6y = 0$$

⋮

done in the previous example.

$$y_h = C_1 e^{-2x} + C_2 e^{-3x}$$

Find y_p

Stare at $K(x)$:-

- Check if $K(x) = 3x$ is a solution to the homogenious part.
- No.

$$y_p = a_0 + a_1 x$$

- Find a_0, a_1
- We do that by substitution.

$$y_p' = a_1$$

$$y_p'' = 0$$

plug in the main equation:-

$$0 + 5a_1 + 6(a_0 + a_1 x) = 3x$$

$$\overbrace{5a_1 + 6a_0}^{\text{Constant}} + \overbrace{6a_1 x}^{x\text{-term}} = 3x + 0$$

from equating both sides:-

$$5a_1 + 6a_0 = 0$$

$$6a_1 x = x \Rightarrow a_1 = 1/2$$

$$\frac{5}{2} + 6a_0 = 0 \Rightarrow a_0 = -5/12$$

$y_g = y_h + y_p$

$= C_1 e^{-2x} + C_2 e^{-3x} + \frac{-5}{12} + \frac{1}{2}x$

Ex:- $y^{(2)} + 5y' + 6y = 2e^{3x}$

For homogenious part is done.

$y_h = C_1 e^{-2x} + C_2 e^{+3x}$

Find y_p : $\Rightarrow$ $y_p = a_0 e^{3x}$

$y'_p = 3a_0 e^{3x}$

$y''_p = 9a_0 e^{3x}$

$9a_0 e^{3x} + 15a_0 e^{3x} + 6a_0 e^{3x} = 2e^{3x}$

$(9a_0 + 15a_0 + 6a_0)e^{3x} = 2e^{3x}$

$30a_0 = 2 \Rightarrow a_0 = 1/15$

$y_p = (1/15)e^{3x}$

$y_g = y_h + y_p$

$= C_1 e^{-2x} + C_2 e^{-3x} + \frac{1}{15}e^{3x}$

Subject: (Date / / :التاريخ) :الموضوع

Sunday 22/3/2009

UCM :-

$y'' + 9y = 2\sin(5x)$

Solution:

Solve the associated homogeneous system.

$y'' + 9y = 0$

$y = e^{mx}$

Char (DE) = $m^2 + 9$

$m^2 + 9 = 0$

$m = \pm 3i$

$y_1 = \cos(3x)$

$y_2 = \sin(3x)$

$y_h = C_1 \sin(3x) + C_2 \cos(3x)$

Now find y_p :

look at $K(x) = 2\sin(5x)$ "Not a solution to the homogenous"

$y_p = a_0 \sin(5x) + a_1 \cos(5x)$

$y_p' = 5a_0 \cos(5x) - a_1 (5) \sin(5x)$

$y_p'' = -25a_0 \sin(5x) - 25a_1 \cos(5x)$

$-25a_0 \sin(5x) - 25a_1 \cos(5x) + 9a_0 \sin(5x) + 9a_1 \cos(5x) = 2\sin(5$

$-16a_0 \sin(5x) - 16a_1 \cos(5x) = 0 + 2\sin(5x)$

$-16a_0 = 2 \Rightarrow a_0 = -1/8$

$-16a_1 = 0 \Rightarrow a_1 = 0$

$y_p = (-1/8)\sin(5x)$

$y_g = y_h + y_p$

$= C_1 \sin(3x) + C_2 \cos(3x) - \frac{1}{8}\sin(5x)$

$y'' + 9y = 2\sin(3x)$

Solution:-

$y_h = C_1 \cos(3x) + C_2 \sin(3x)$

look @ $K(x) = 2\sin(3x)$ → solution to the homogenous.

use product Rule.

Here $y_p = [a_0 \sin(3x) + a_1 \cos(3x)]\, x$

⋮

$y' + 2y = e^{-2x} + x$

Slove the associated homogenous differential equation:-

$y' + 2y = 0$

$y = e^{mx}$

$m + 2 = 0 \Rightarrow m = -2$

$y_1 = e^{-2x}$

$y_h = e^{-2x}(C_1)$

To find y_p :-

$K(x) = e^{-2x} + x$

→ Solution to the homogenous.

$y_p = x(a_0 e^{-2x}) + a_1 + a_2(x)$

Subject: (Date / / :التاريخ) :الموضوع

$$y_p' = a_0 e^{-2x} + -2a_0 x e^{-2x} + a_2$$

$$a_0 e^{-2x} + -2a_0 \cancel{x e^{-2x}} + a_2 + 2a_0 \cancel{x e^{-2x}} + 2a_1 + 2a_2 x = e^{-2x} + x$$

$$a_0 e^{-2x} + 2a_2 x + 2a_1 + a_2 = e^{-2x} + x + 0$$

$a_0 = 1$

$2a_2 = 1 \Rightarrow a_2 = 1/2$

$2a_1 + a_2 = 0$

$2a_1 + \frac{1}{2} = 0$

$a_1 = \frac{-1}{4}$

$$y_p = x e^{-2x} + -1/4 + (1/2)x$$

$$y_g = y_h + y_p$$

$= \cancel{c_1 \cos x}$

$= c_1 e^{-2x} + x e^{-2x} + \frac{-1}{4} + (1/2)x$

what would happen if we have conditions, 24/3/2009 Tuesday

* Variation Method :-

→ Synthetic Division :-

Solve :-

(*) $2x^3 - 7x^2 - 5x + 4 = 0$, $x = -1$ is a solution.

To get the other two roots :-

-1	2	-7	-5	4
+	0	-2	9	-4
	2	-9	4	0

always zero

Co-effecient x^2 — co-effecient of x — Constant

→ Quotient when we ~~we solve~~ divide by $x+1$

$2x^2 - 9x + 4 = 0$

$(2x-1)(x-4) = 0$

$x = 1/2$

$x = 4$

Ex:- Solve :- $2y^{(3)} - 7y^{(2)} - 5y + 4 = 0$, given $y = e^{4x}$

$y = e^{mx}$

Char (D.E) $= 2m^3 - 7m^2 - 5m + 4 = 0$

Since $y = e^{4x}$ is a solution $\Rightarrow m = 4$

4	2	-7	-5	4
	0	8	4	-4
	2	1	-1	0

Note :-

If there is a missing term we put zero in it's place.

$2m^2 + m - 1 = 0$

$(2m - 1)(m + 1) = 0$

$m = 1/2 \qquad y_2 = e^{1/2\, x}$

$m = -1 \qquad y_3 = e^{-x}$

$$y_g = C_1 e^{4x} + C_2 e^{1/2\, x} + C_3 e^{-x}$$

H.W :-

$y^{(4)} - 2y^{(3)} + 2y^{(2)} - 4y' = 0 \qquad \left(y^{(4)} - 2y^{(3)} + 2y^{(2)} - 4y' = 0 \right)$

Given $y = e^{2x}$ is a solution.

* Variation Method :- ("Used to find y_p")

Result:- "True in General", (Practice for order Two)

$$a_2(x)\, y^{(2)} + a_1(x)\, y' + a_0(x)\, y = K(x)$$

- we Know: $y_g = y_h + y_p$

- Variation used to find y_p
- Assume y_1, y_2 are two independent solution to the associated homogeneous system.

- There are two functions $f_1(x), f_2(x)$ such that

$$y_p = f_1(x)\, y_1 + f_2(x)\, y_2$$

- To find f_1, f_2

do:-

$$f_1'(x)\, y_1 + f_2'(x)\, y_2 = 0$$

$$f_1'(x)\, y_1' + f_2'(x)\, y_2' = \frac{K(x)}{a_2(x)}$$

$a_2(x)$ → the co-effecient of $y^{(2)}$

- Solve (C-Reman) for f_1', f_2'

$$f_1 = \int f_1'\, dx$$

$$f_2 = \int f_2'\, dx.$$

Subject: (Date / / :التاريخ) :الموضوع

Example :-

$$2y^{(2)} + y' = \ln(x)$$

$a_2(x)$

Solution :-

Solve the associated homogenous system.

~~Char (D.E):~~

$2y^{(2)} + y' = 0$

$y = e^{mx}$

Char (D.E) :

$2m^2 + m = 0$

$m = 0 \qquad y_1 = 1$

$m = -1/2 \qquad y_2 = e^{-1/2\, x}$

$$y_h = C_1 + C_2 e^{-1/2\, x}$$

$y_p = f_1 y_1 + f_2 y_2$

$= f_1 \cdot (1) + f_2 \cdot (e^{-1/2\, x})$

To find f_1, f_2 :-

$f_1'(x) \cdot 1 + f_2' e^{-1/2\, x} = 0$

$f_1(x)(0) + f_2'(-\frac{1}{2} e^{-1/2\, x}) = \frac{\ln(x)}{2}$

$f_2'(x) = \frac{\ln(x)}{2 e^{-1/2\, x}} \cdot x - 2$

$f_2'(x) = \frac{-\ln(x)}{e^{-\frac{1}{2}x}} = -e^{(1/2)x} \cdot \ln(x)$

$f_2(x) = \int -e^{(1/2)(x)} \cdot \ln(x)\, dx$

$f_1'(x) + f_2'(x) e^{-1/2\, x} = 0$

$f_1'(x) + -e^{\frac{1}{2}x} \ln(x)\, e^{-1/2\, x} = 0$

$f_1'(x) = \ln(x)$

$\int \ln(x) =$

$\int \underset{dv}{1} \cdot \underset{u}{\ln(x)} =$

$\underset{u}{x}\, \underset{v}{\ln(x)} - \int \underset{v \cdot du}{1\, dx}$

$v = x \qquad du = \frac{1}{x}$

$dv = 1 \qquad u = \ln(x)$

$\Rightarrow x\ln(x) - x$

"Here we don't add the constant to the result of the integration since the constant is already included in the solution".

$y_p = (x\ln(x) - x)\, 1 + \left[\int -e^{-1/2\, x} \ln(x)\, dx\right] e^{-1/2\, x}$

$$y_g = \underbrace{C_1 + C_2 e^{-1/2\, x}}_{y_h} + y_p$$

Subject: (Date / / :التاريخ) :الموضوع

29/3/2009 Sunday

Variation Method :-

Solve :.

$y^{(4)} + y^{(2)} = x$ Note :- $a_n = a_4 = 1$

Solution :-

$y^{(4)} + y^{(2)} = 0$

$y = e^{mx}$

* Char (D.E) : $m^4 + m^2 = 0$

$m^2(m^2+1) = 0$

$m = 0 \longrightarrow y_1 = e^{0x} = 1$

$m = 0 \longrightarrow y_2 = xe^{0x} = x(1) = x$

$m = 0 \pm i$ — $y_3 = \sin(x)$, $y_4 = \cos(x)$

* $y_h = C_1 + C_2 x + C_3 \sin(x) + C_4 \cos(x)$

* $y_p = (f_1).(1) + (f_2).(x) + (f_3).(\sin(x)) + (f_4).(\cos(x))$

* To find f_1, f_2, f_3, f_4

$f_1'.1 + f_2'.x + f_3'.\sin(x) + f_4'.\cos(x) = 0$

$f_1'.(0) + f_2'.1 + f_3'\cos(x) + (f_4')(-\sin(x)) = 0$

$0 + 0 + (f_3')(-\sin(x)) + (f_4')(-\cos(x)) = 0$

$0 + 0 + (f_3')(-\cos(x)) + (f_4')(\sin(x)) = \frac{x}{1}$

Suppose

$k(x) = x$

$m=0$ $y_1 = x$ $y_{p} = ?$

$m=0$ $y_2 = x^2$

Subject: (Date / / :التاريخ) :الموضوع

29/3/2009 Sunday

Variation Method :-

Solve :.

$$y^{(4)} + y^{(2)} = x \qquad \text{Note :- } a_n = a_4 = 1$$

Solution :-

$$y^{(4)} + y^{(2)} = 0$$

$$y = e^{mx}$$

* Char (D.E) $: m^4 + m^2 = 0$

$$m^2(m^2+1) = 0$$

$$m = 0 \longrightarrow y_1 = e^{0x} = 1$$

$$m = 0 \longrightarrow y_2 = xe^{0x} = x(1) = x$$

$$m = 0 \pm i \; \begin{cases} y_3 = \sin(x) \\ y_4 = \cos(x) \end{cases}$$

* $y_h = c_1 + C_2 x + C_3 \sin(x) + C_4 \cos(x)$

* $y_p = (f_1).(1) + (f_2)(x) + (f_3).(\sin(x)) + (f_4)(\cos(x))$

* To find f_1, f_2, f_3, f_4

$$f_1' . 1 + f_2' . x + f_3' . \sin(x) + f_4' . \cos(x) = 0$$

$$f_1' . (0) + f_2' . 1 + f_3' \cos(x) + (f_4')(-\sin(x)) = 0$$

$$0 + 0 + (f_3')(-\sin(x)) + (f_4')(-\cos(x)) = 0$$

$$0 + 0 + (f_3')(-\cos(x)) + (f_4')(\sin(x)) = \frac{x}{1}$$

Suppose

$m=0 \quad y_1 = x \qquad k(x) = x$

$m=0 \quad y_2 = x^2 \qquad y_p = ?$

$y^{(4)} + y^{(2)} = x$

Solve using U.C.M :-

$m^4 + m^2 = 0 \Rightarrow m^2(m^2+1) = 0$

$m = 0 \rightarrow y_1 = 1$

$m = 0 \rightarrow y_2 = x$

$m = 0 \pm i \rightarrow y_3 = \sin(x)$

$y_4 = \cos(x)$

$y_h = C_1 + C_2 x + C_3 \sin(x) + C_4 \cos(x)$

$y_p = x^2(a_0 + a_1 x) = a_0 x^2 + a_1 x^3$ " we multiplied by x^2 bcz of having m^2 "

$y_p' = 2a_0 x + 3a_1 x^2$

$y_p'' = 2a_0 + 6a_1 x$

$y_p''' = 6a_1$

$y_p^{(4)} = 0$

$0 + 2a_0 + 6a_1 x = 0 + x \Rightarrow a_1 = 1/6 , \ a_0 = 0$

$y_p = x^2(0 + 1/6\, x) = \frac{1}{6} x^3$.

Example:- Solve:-

$y^{(2)} - 2y' + y = e^x$

Char D.E :-

$m^2 - 2m + 1 = e^x$

$(m-1)^2 = e^x$

$m - 1 = 0 \Rightarrow y_1 = e^x$

$m - 1 = 0 \Rightarrow m = 1 \Rightarrow y_2 = x e^x$

$y_h = C_1 e^x + C_2 x e^x$

$y_p = \underline{\underline{x^2}}(a_0 e^x)$

↳ Be careful !!

Chapter One :-

1.st order D.E :-

$$a_1(x)\,y' + a_0(x)\,y = K(x) \quad , \quad a_1(x) \neq 0$$

Trick:- rewrite in the standard form. (make coof of $y' = 1$)

$$y' + \underbrace{\frac{a_0(x)}{a_1(x)}}_{q(x)}\, y = \underbrace{\frac{K(x)}{a_1(x)}}_{F(x)}$$

$y' + q(x)\,y = F(x) \longrightarrow$ standard form

Integral factor :- $I = e^{\int q(x)\,dx}$

* by Integral factor:-

$$e^{\int q(x)dx}\, y' + e^{\int q(x)dx}\, q(x)\, y = e^{\int q(x)dx} \cdot F(x)$$

$$\left(e^{\int q(x)dx}\, y\right)' = e^{\int q(x)dx} \cdot F(x)$$

now integrate both sides :-

$$e^{\int q(x)dx} \cdot y = \int e^{\int q(x)dx} \cdot F(x)\, dx$$

Solve for y :

$$y = \frac{\int e^{\int q(x)dx} \cdot F(x)\, dx}{e^{\int q(x)\,dx}}$$

(b) assume $y(0) = 2$ Find y.

$$y = \frac{3}{2}x^2 + Ce^{-x^2} - 3/2$$

$$y(0) = \frac{3}{2}(0)^2 + Ce^0 - 3/2$$

$$2 = C - \frac{3}{2} \Rightarrow C = 7/2 \approx 3.5$$

$$y = (3/2)x^2 + 3.5e^{-x^2} - 3/2$$

(C) Find the largest interval around (x) for part (b)

it can't be done under part bc condition since the function is not continuous at 0. we have to pick another value.

Example :-

$$(x+1)y' + y = x+4$$

Solution :-

$$y' + \frac{y}{x+1} = \frac{x+4}{x+1}$$

$$Q(x) = \frac{1}{x+1} \Rightarrow I = e^{\int \frac{1}{x+1}dx} = e^{Ln(x+1)} = x+1$$

$$y = \frac{\int F(x) I \, dx}{I} = \frac{\int \frac{(x+4)}{x+1}(x+1)\,dx}{x+1}$$

$$= \frac{\frac{x^2}{2} + 4x + C}{x+1} = \frac{1}{x+1}\left[\frac{x^2}{2} + 4x + C\right]$$

Example:-

$$y' + \cos(x)\cdot y = \cos(x)$$

$$Q(x) = \cos(x) \qquad F(x) = \cos(x)$$

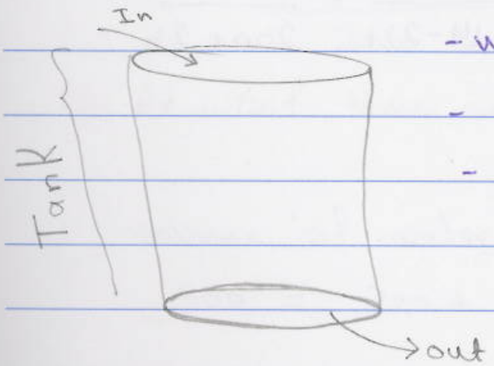
$$I = e^{\int Q(x)\,dx} = e^{\int \cos(x)\,dx} = e^{\sin(x)}$$

$$y = \frac{\int \cos(x)\, e^{\sin(x)}\,dx}{e^{\sin(x)}}$$

$$u = \sin(x) \Rightarrow du = \cos(x)\,dx$$

$$y = \frac{e^{\sin(x)} + C}{e^{\sin(x)}} = 1 + Ce^{-\sin(x)}$$

5/4/2009 Sunday



- water Tank.
- mixture
- Capacity of the tank = 500 gallons.
- Rate in :- 4 gal/min (four), and each gallon has 2 pound of salt.

- Rate out :- 2 gal/min

Find the amount of salt in the tank at any time (t)

Assume at t=0, the amount of salt in the tank is 10 pounds.

Assume the tank has 200 gallon of water at t=0.
(mixture)

Solution :-

$A(t)$ = Amount of salt in the tank at time t.

Given :-

rate change of salt in the tank

$$\frac{dA}{dt} = \underbrace{4 * 2}_{\text{in-Salt per minute.}} - \underbrace{? * 2}_{\text{out-Salt per minute}}$$

$?$ = Concentration of salt in the tank.

$$\text{Concentration per gallon} = \frac{A(t)}{\text{Volume of water}}$$

Volume of water = initial amount of water $\pm$ $(R_{in} - R_{out})\,t$

$$\text{Concentration} = C(t) = \frac{A(t)}{V_0 + (R_{in}-R_{out})t} = \frac{A(t)}{200 + (4-2)t} = \frac{A(t)}{200+2t}$$

$$\frac{dA}{dt} = 8 - C(t)\cdot(2)$$

$$\frac{dA}{dt} = 8 - \frac{A(t)}{200+2t} * 2$$

$$\frac{dA}{dt} + \frac{2}{200+2t} A(t) = 8$$

1.st order linear D.E

Find $A(t)$ = ------ in terms of t.

$$A(t) = \int F(t) \cdot I \, dt \Big/ I$$

$$I = e^{\int \frac{2}{200+2t} dt} = e^{Ln(200+2t)} = 200 + 2t$$

$$A(t) = \frac{\int 8 * (200+2t) \, dt}{200+2t} = \frac{\int 1600 + 16t \, dt}{200+2t}$$

$$A(t) = \frac{1600t + 8t^2 + c}{200+2t}$$

$$A(0) = 10$$

$$10 = \frac{0 + c}{200} \Rightarrow c = 2000$$

$$A(t) = \frac{1600t + 8t^2 + 2000}{200+2t}$$

(B) At what time an over flow will occur?

volume of water at the tank = initial volume + (Rin − Rout) t

$$500 = 200 + 2t \Rightarrow t = 150$$

(C) What is the amount of salt in the tank at t = 10 min.

$$A(t) = (1600(10) + 8(10)^2 + 2000) \Big/ 200 + 2(10)$$

Ex:-

Cooling - Warming Problem :-

It includes two thing(s):-

- 1.st :- Temprature change with time
- 2.nd :- The Temprature is constant.

Outside temprature is 24 C°; The temp of the engine of your car is 400 °C, when you stopped the engine. Find the temprature of the engine at any time (t).

Solution:-

$T(t)$ = Temp of engine at any time (t)

$$\frac{dT}{dt} = \alpha (T - T_0)$$

T_0 → The constant Temp "outside Temp".

α → Constant we calculate

$$\frac{dT}{dt} - \alpha T = -\alpha T_0$$

For water Tank application:-

21, 23, 25, 27, page 91 (3.1)

For Cooling - Warming

13, 17, 19, 15 page 90 (3.1)

Problem 14 :- page 90 :-

* A thermometer is taken from inside room to the outside, where the air temp is 5°F. After 1 minute the thermometer read 55° F, and after 5 minutes it read 30° F. What is the initial temp of the inside room?

Solution :-

$T(t)$ = Temp of Thermometer at any time (t)

T_0 = Temp that stays constant. "Here is the outside temp".

* $$\frac{dT}{dt} = \alpha(T - T_0)$$ Newton's Law.

* The initial temp of the inside room $\equiv T(t=0)$

* We have to find $T(t)$

$$\frac{dT}{dt} = \alpha(T-5) \Rightarrow \frac{dT}{dt} - \alpha T = -5\alpha$$

$$I = e^{\int Q(t)\,dt} = e^{\int -\alpha\,dt} = e^{-\alpha t}$$

$$T = \int F(t) \cdot I\,dt \,/\, I$$

$$= \int -5\alpha e^{-\alpha t}\,dt \,/\, e^{-\alpha t} = \frac{5e^{-\alpha t} + C}{e^{-\alpha t}}$$

$$T(t) = 5 + Ce^{\alpha t}$$

Subject: (Date / / :التاريخ) :الموضوع

$T(1) = 5 + Ce^{\alpha(1)} = 55$

$5 + Ce^{\alpha} = 55 \Rightarrow (1)$

$T(5) = 5 + Ce^{\alpha(5)} = 30$

$5 + Ce^{5\alpha} = 30 \longrightarrow (2)$

from (1) $Ce^{\alpha} = 50 \Rightarrow (a)$

from (2) $Ce^{5\alpha} = 25 \longrightarrow (b)$

$a/b \Rightarrow \dfrac{Ce^{\alpha}}{Ce^{5\alpha}} = \dfrac{50}{25} \Rightarrow e^{-4\alpha} = 2 \Rightarrow -4\alpha \ln e = \ln 2$

$\Rightarrow \ln(2) = -4\alpha \Rightarrow \alpha = \ln(2)/-4$

$50 = Ce^{\alpha} \Rightarrow 50 = C e^{-\frac{1}{4}\ln(2)}$

$50 = C e^{\ln\left(\frac{1}{\sqrt[4]{2}}\right)} \Rightarrow 50 = c/\sqrt[4]{2}$

$C = 50 * \sqrt[4]{2}$

Now, Temp inside room :-

$T(0) = 5 + 50\sqrt[4]{2}\, e^{0}$

$= 5 + 50\sqrt[4]{2}$

Growth _ Decay :- "Growth Example"

rate of growth in

* Population in Shj is propotional to P(t) at any time t.

* P(t) Population at time t.

If population of sharjah is 400,000 now, and rate of growth of population in shj is 0.002 of P(t).

(1) What will be the population of shj within 4 years from now.

(2) How long would it take to double the population.

P(t) = Population of shj at any time (t).

$$\frac{dP}{dt} = \alpha P(t)$$

$$= 0.002 P(t)$$

$$\frac{dP}{dt} \underbrace{- 0.002}_{Q(t)} P = \underbrace{0}_{F(t)}$$

$$I = e^{\int Q(t)\,dt} = e^{\int -0.002\,dt} = e^{-0.002t}$$

$$P(t) = \frac{\int I . F\,dt}{I} = \int 0\, I / I = C / I = Ce^{0.002t}$$

$$P(0) = Ce^0 = 400\,000$$

$$C = 400\,000$$

$$P(t) = 400000\, e^{0.002t}$$

* let t = 4

$$P(4) = 400\,000\, e^{0.002(4)}$$

* Time to double:-

$$800\,000 = 400\,000\, e^{0.002t}$$

Subject: (Date / / التاريخ:) الموضوع:

12/4/2009 Sunday

Decay...

Ex:-

Chemical called "A", after 15 years it's determined that 0.043% of the initial amount A_0 has disintegrated. Find the half life of this chemical. Given, the rate of Decay is propotional to the amount remaining (present amount)

Solution:-

$A(t)$ = The remaining amount of chemical after t years
or the present amount of chemical after t years.

$$\frac{dA}{dt} = \alpha . A(t)$$

" Since we are dealing with decay, α is a negative
In case of growth, α is positive

$A(t) = A_0$ at $t=0$

$$A(15) = A_0 - \frac{0.043}{100} A_0$$

$$= A_0\left(1 - \frac{0.043}{100}\right)$$

$$= A_0(0.99957)$$

$$\frac{dA}{dt} - \alpha A = 0$$

$A = e^{mt}$

Char(D.E) = $m - \alpha = 0$

$m = \alpha$

$y = e^{\alpha t}$

Subject: (Date / / :التاريخ) :الموضوع

$$y_g = C_1 e^{\alpha t} = A(t)$$

To find C_1:

we know $A(0) = A_0$

$$A_0 = C_1 e^{\alpha(0)} = C_1$$

$$A(t) = A_0 e^{\alpha t}$$

but $A(15) = 0.99957$

To find α we use $A(15) = 0.9995\cancel{7}\, A_0$.

$$A(15) = \cancel{A_0} e^{\alpha 15} = 0.99957 \cancel{A_0}$$

$$e^{15\alpha} = 0.99957$$

$$15\alpha = \ln(0.99957) \Rightarrow \alpha = \ln(0.99957)/15$$

$$\alpha = -2.87 \times 10^{-15}$$

$$= -0.0000287$$

$$A(t) = A_0 e^{-2.87 \times 10^{-15} t}$$

To find half life of the chemical:

$$\tfrac{1}{2}\cancel{A_0} = \cancel{A_0} e^{-2.87 \times 10^{-15}}$$

$$\ln 0.5 = -2.87 \times 10^{-5} t \Rightarrow t = \ln(0.5) / -2.87 \times 10^{-15}$$

$$\cancel{= 2.415 \times 10^{14}}$$

Note:-

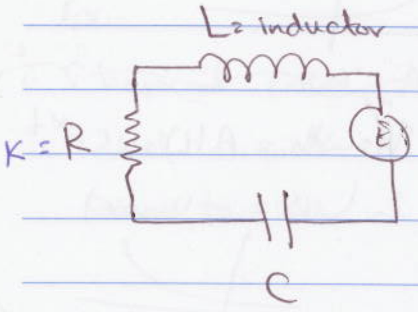
decay ~~growth~~ $= A(t) = c e^{\alpha t}$

$(\alpha = -ve\ \#)$

growth $= A(t) = c e^{\alpha t}$

$(\alpha = +ve\ \#)$

Chapter Five



Charge on the capacitor = $q(t)$ = amount of charge on capacitor at any time.

$$i(t) = \frac{dq}{dt}$$

$$\frac{q}{c} + \frac{k\,dq}{dt} + L\frac{di}{dt} = E$$

$$\frac{q}{c} + \frac{k\,dq}{dt} + L\frac{d^2q}{dt^2} = E$$

(E, L, k, C) are constant.

"Our goal find $q(t)$".

$$i(t) = \frac{dq}{dt}$$

$$i(t) = \frac{d^2q}{dt}$$

Subject: (Date / / التاريخ) وضوع:

Ex:-

$L = 0.25\ H,\quad R = 10\ \Omega,\quad C = 0.001\ F,\quad E(t) = 0$

$q(0) = q_0,\quad i(0) = 0$

$$i(0) = 0 = \left.\frac{dq}{dt}\right|_{t=0}$$

$$1000\, q(t) + 10\frac{dq}{dt} + 0.25\frac{d^2q}{dt^2} = 0$$

$q(t) = e^{mt}$

Char (D.E)

$0.25\, m^2 + 10\, m + 1000 = 0$

$$m = \frac{-10 \pm \sqrt{100 - 1000}}{0.5}$$

$$m = \frac{-10 \pm 30 i}{0.5} = -20 \pm 60 i$$

$m = -20 + 60 i$

$q_1(t) = e^{-20t}\sin(60t)$

$q_2(t) = e^{-20t}\cos(60t)$

$q(t) = C_1 e^{-20t}\sin(60t) + C_2 e^{-20t}\cos(60t)$

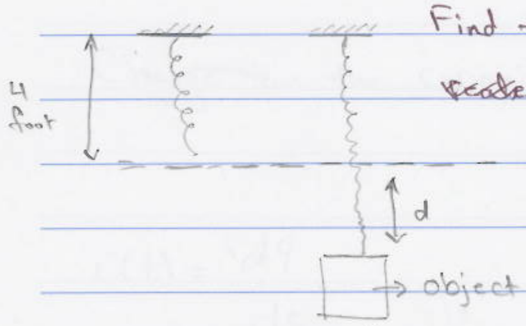
Subject: (Date / / :التاريخ) :الموضوع

14/4/2009 Tuesday

Applications:-

Problem (22):- Page 196:-

A 4-foot spring measure 8 foot after a mass weighting 8 pounds is attached to it. The medium through which the mass moves offers a force numericalley equal to $\sqrt{2}$ times the instant velocity.



Find the equation of the motion if the mass is initialy ~~reale~~ released from the equilb position with a downward velocity of 5 foot/s.

d is the distance (always)

This state is equilb position.

K = spring Costant.

Weight = Distance * K

Distance = 8 - 4 = 4 foot

K = 8 pound / 4 foot
= 2

x(t) = the motion of spring at any time t.

We need to find the motion, ~~of the spring at any time t~~

Motion = x(t)

Notes

"Distance = how much it streched"

In case using other units:-

weight = pounds
length = foot
gravity = 32 ft/s²

1 ft = 12 inches

weight = Kg
length = m
gravity = 9.8 m/s²

Subject: (Date / / التاريخ:) الموضوع:

* The surrounding (medium) offers a force $= \sqrt{2} \cdot V_0$
$= \sqrt{2} \cdot \frac{dx}{dt}$

* Released from equilberuim $\longrightarrow x(0) = 0$
* Released above equilb (ie 2 foot above equilb) $\rightarrow x(0) = -2$
* Released below equilb (ie 2 ft below equilb) $\longrightarrow x(0) = +2$

in the question:
* released from equilb position with downward $V = 5$ ft/s

$x'(0) = 5$

$$\boxed{F = \beta V}$$

$F = \beta \frac{dx}{dt}$ in our case $\beta = \sqrt{2}$

$$\boxed{\frac{d^2x}{dt^2} + \frac{\beta}{m} \cdot \frac{dx}{dt} + \frac{K}{m} x = \frac{F(t)}{m}}$$

$F(t) =$ external driven force.
$\frac{d^2x}{d^2t} =$ acceleration

Our Goal:
Find, $K, m, \beta, F(t)$

mass $= m =$ weight$/g = w/32$
$m = 8/32 = 0.25 = 1/4$
$K = 2$
$\beta = \sqrt{2}$
$F(t) = 0$

$$x''(t) + \frac{\beta}{m} x'(t) + \frac{k}{m} x(t) = 0$$

$$x''(t) + 4\sqrt{2}\, x'(t) + 8x(t) = 0$$

Subject: (Date / / :التاريخ) :الموضوع

initial Condition:-

$x(0) = 0$

$\dot{x}(0) = 5$

To solve for $x(t)$:-

$x = e^{mt}$

Char (D.E) $= m^2 + 4\sqrt{2}\,m + 8 = 0$

$m = -4\sqrt{2} \pm \sqrt{32 - 32} \Big/ 2$

$= -2\sqrt{2}$

$m = -2\sqrt{2} \rightarrow x = e^{-2\sqrt{2}\,t}$

$\rightarrow x = t\,e^{-2\sqrt{2}\,t}$

$x(t) = C_1 e^{-2\sqrt{2}t} + C_2 t e^{-2\sqrt{2}t}$

$x(0) = C_1 e^0 + 0 = 0 \Rightarrow C_1 = 0$

$x(t) = C_2 t e^{-2\sqrt{2}t}$

$x'(t) = C_2 e^{-2\sqrt{2}t} + -2\sqrt{2}\,t\,e^{-2\sqrt{2}t}\,(C_2)$

$= C_2 \left(e^{-2\sqrt{2}t} + -2\sqrt{2}\,t\,e^{-2\sqrt{2}t} \right)$

$x'(0) = C_2 = 5$

$x(t) = 5\,t\,e^{-2\sqrt{2}t}$

Subject: (Date / / التاريخ) الموضوع:

* motion of Spring so

suppose the following equation is motion equation :-

$x(t) = 3\sin(5t) + 10\cos(5t)$

(1) Amplitude $= \sqrt{(\text{coffecient of } \sin)^2 + (\text{coffecient of } \cos)^2} = \sqrt{3^2 + 10^2}$

(2) Frequency $= 1/\text{Period} =$ # of cycles Per seconds.

(3) Period (in our example : $\frac{2\pi}{5}$

Tuesday 21/4/2009

1st Order non-Linear D.E (2.5)

Linear :- "y" and all its derivative are raised to power one.

Standard form of Bernoulli :-

$$y' + a_0(x)\,y = K(x)\,y^n \quad , n \neq 1 \quad , n \in R$$

if $n = 1 \Rightarrow y' + (a_0(x) - K(x))y = 0$

(1) make an Algebra Trick

(2) Reduce it to 1st order linear in terms of u , x , where

u is the new dependent variable.

Trick :-

let $u = y^{1-n}$ "y: dependent"

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$

$$\frac{du}{dx} = (1-n)\,y^{-n} \cdot \frac{dy}{dx} = \frac{1-n}{y^n} \cdot \frac{dy}{dx} \rightarrow y'$$

now Solve for y'

Subject: (Date / / :التاريخ) :الموضوع

Continue ...

$$y' = \left(y^n/_{1-n}\right) * \left(\frac{du}{dx}\right)$$

u'

given

now look at the equation :.

$$y' + a_0(x) y = K(x) y^n$$

sub for y' :.

$$\frac{y^n}{1-n} u'(x) + a_0(x) y = K(x) y^n \quad \text{"Divide by } \frac{y^n}{1-n}\text{"}$$

$$u'(x) + (1-n)\, a_0(x) \frac{y}{y^n} = (1-n) K(x)$$

$$\frac{y}{y^n} = y^{1-n} = u$$

$$u'(x) + (1-n)\, a_0(x)\, u = (1-n) K(x)$$

To write the solution $y = u^{\frac{1}{1-n}}$

$$\left(y^{1-n}\right)^{\frac{1}{1-n}} = u^{\frac{1}{1-n}}$$

$$y = u^{\frac{1}{1-n}}$$

Example :.

$$y' + \frac{1}{1+2x} y = \frac{x^3}{y}$$

$$y' + \frac{1}{1+2x} y = y^{-1} x^3$$

$a_0(x)$ $K(x)$

$n = -1$

$u = y^{1-n} = y^2$, $1-n = 1-(-1) = 2$

rewrite the above in terms of u and x

$$u' + (1-n)\, a_0(x)\, u = (1-n)\, K(x)$$

$$u' + 2 \cdot \frac{1}{1+2x} u = 2x^3$$

$$I = e^{\int Q(x)\,dx} = e^{\int \frac{2}{1+2x}dx} = e^{\ln|1+2x|} = 1+2x$$

$$u = \frac{\int F(x)\cdot I\,dx}{I}$$

Note:-
$e^{3\ln(2x)}$
$= e^{\ln(2x)^3}$
$= (2x)^3$

$$= \frac{\int 2x^3(1+2x)\,dx}{1+2x} = \frac{\frac{1}{2}x^4 + \frac{4}{5}x^5 + C}{1+2x}$$

$$y = \sqrt{u} = u^{1/2}$$

$$= \sqrt{\frac{\frac{x^4}{2} + (4/5)x^5 + C}{1+2x}}$$

Example :-

$$xy' + y = \cancel{x + x^2y^2/y} \quad \frac{x + x^2y^2}{y}$$

$$xy' + y = \frac{x}{y} + x^2y$$

$a_0(x)$ $$xy' + y - x^2y = xy^{-1}$$

$$xy' + (1-x^2)y = xy^{-1} \quad K(x)$$

$$y' + \left(\frac{1-x^2}{x}\right)y = 1y^{-1} \quad , n = -1$$

$$u = y^{1-n} = y^2 \quad , n = -1$$

rewrite ~~Q(x)~~ in term of u & x

$$u' + \underbrace{2\frac{(1-x^2)}{x}}_{Q(x)}u = 2$$

$$I = e^{\int 2(1-x^2)/x} = e^{\ln(x^2) - x^2} = e^{\ln(x^2)} \cdot e^{-x^2}$$

Subject: (Date / / التاريخ:) الموضوع:

Thursday 23/4/2009

1.St Order non-Linear "Exact"

* we have to remember implicit diff:-

$xy + ye^x + x^3 = 13 \rightarrow (1)$

$y + xy' + ye^x + e^x y' + 3x^2 = 0$

$y' = -y - 3x^2 - ye^x / x + e^x \Big] \rightarrow (2)$

* in 205 (1) will be calculated as (2) is given.

* We have to know the parial derivative:-

$F(x,y) = xy + ye^x + x^3$

$F_x = y + ye^x + 3x^2$ "Derive w.r.t x" (treat y as constant).

$F_y = x + e^x + 0$

F_{xy} = Derive F_x with respect to y (treat x as constant)

$= 1 + e^x$

$F_{yx} = 1 + e^x$

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

$= -y - ye^x - 3x^2 / x + e^x$

→ We will be given dy/dx

→ we will find $F(x,y)$

→ We will write $F(x,y) = C$

Another notation :-

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

$$F_y\,dy = -F_x\,dx$$

$$\boxed{F_y\,dy + F_x\,dx = 0}$$ standard form

Example :-

$$\underbrace{(x^3 + 2y)}_{F_y}\,dy \underbrace{- (-3x^2y + x)}_{F_x}\,dx = 0$$

F_y $\quad$ F_x "including the -ve sign"

Check :-

$$F_{xy} = F_{yx} \xrightarrow{\text{Yes}} \text{Use exact}$$

$$3x^2 = 3x^2$$

We can use exact.

* now pick any one :- "either dy or dx"
* Then integrate it :-

$$\int F_y\,dy \Rightarrow \int F_y\,dy = F(x,y)$$

$$\int x^3 + 2y\,dy = x^3y + y^2 + \underline{\underline{K(x)}}$$ work as constant since the function in term of dy

Subject: (Date / / :التاريخ) الموضوع:

26/4/2009 Exact Sunday

1.St Order non-Linear

Ex:- Solve

$$\frac{dy}{dx} = \frac{e^x y + 3y^2 + x}{-e^x + -3x + 10}$$

Step :-

① rewrite in standard form.

$$(-e^x + -3x + 10)\,dy + -(e^x y + 3y^2 + x)\,dx = 0$$

Derive w.r.t x — F_y — F_x — Derive w.r.t y

② Check $F_{yx} = F_{xy}$??

$$-e^x - 3 = -e^x - 3$$

③ Apply exact

- find a function $f(x,y)$
- Solution $f(x,y) = C$
- integrate F_y or F_x

$$\int F_y\,dy = \int (-e^x - 3x + 10)\,dy = \underbrace{-e^x y - 3xy + 10y + K(x)}_{F(x,y)}$$

- To find $K(x)$, we use the un integrated F'
 "if $\int F_y\,dy \rightarrow$ use F_x
 $\int F_x\,dx \rightarrow$ use F_y

$F_x =$ Partial derivative of $F(x,y)$

$$= -ye^x - 3y + K'(x)$$

Now equate the calculated diffrentiate f^n with the given f^n

Subject: (Date / / :التاريخ) :الموضوع

$$\underbrace{-\cancel{e^x y} - \cancel{3y} - x}_{\text{Given } F(x)} = \underbrace{-\cancel{y e^x} - \cancel{3y} + K'(x)}_{\text{Calculated } F_x}$$

$$-x = K'(x)$$

Find $K(x)$

$$\int -x \, dx = \int K'(x) \, dx$$

$$-\frac{1}{2} x^2 = K(x)$$

Solution: $\boxed{-e^x y - 3xy + 10y - \frac{1}{2} x^2 = C}$

implicit solution.

Def:- Explicit Solution:-

After we solve the D.E, we solve for y

Implicit Solution:-

Do nothing

By going back to our example:-

$$y = \frac{c + (1/2) x^2}{-e^x - 3x + 10}$$

Subject: (Date / / :التاريخ) الموضــوع:

Seprable

Usually it's 1.st Order (99% non-Linear)

Ex:-

$$(3x+2)\,dx + (y^2+7y)\,dy = 0$$

→ It's a seprable D.E

→ all x-terms with dx

→ all y-terms with dy

now integrate the above equation :

$$\int 3x+2 \, dx + \int y^2 - 7y \, dy = 0$$

$$\frac{3x^2}{2} + 2x + \frac{1}{3}y^3 - \frac{7}{2}y^2 = \int 0 \, dx$$

$$\boxed{\frac{3x^2}{2} + 2x + \frac{1}{3}y^3 - \frac{7}{2}y^2 = C} \longrightarrow \text{implicit.}$$

Solve :

$$x\sqrt{1-y^2}\,dx + \left(y\sqrt{1-x^2}\right)dy = 0$$

Algebra Trick:

Divide by $(\sqrt{1-y^2} * \sqrt{1-x^2})$

$$\frac{x}{\sqrt{1-x^2}}\,dx + \frac{y}{\sqrt{1-y^2}}\,dy = 0$$ (now use seprable)

$$\int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{y}{\sqrt{1-y^2}}\,dy = C$$

$$-\sqrt{1-x^2} + -\sqrt{1-y^2} = C$$

$$\frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}}\,dx + \frac{1}{2}\int \frac{2y}{\sqrt{1-y^2}}\,dy = C$$

Subject: ______ (Date / / التاريخ:) ______ موضـــوع:

$$\int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$\frac{-1}{2}\int -2x\,(1-x^2)^{-1/2}\,dx \qquad u = 1-x^2$$

$$du = -2x\,dx$$

$$= \frac{-1}{2}\int u^{-1/2}\,du$$

$$= -\sqrt{1-x^2}$$

Ex:-

$$\frac{dy}{dx} = \frac{1+y}{3+x}$$

$$(3+x)\,dy + -(1+y)\,dx = 0 \qquad \text{Divide by } (3+x)(1+y)$$

$$\frac{1}{1+y}\,dy + \frac{-1}{3+x}\,dx = 0$$

$$\int \frac{1}{1+y}\,dy + -\int \frac{1}{3+x}\,dx = C$$

$$\ln|1+y| - \ln|3+x| = C$$

Subject: (Date / / التاريخ :) الموضـــوع:

Reduced to Seprable

$$\frac{dy}{dx} = f(ay + bx + c) \quad , \quad a \neq 0$$
$$a, b, c \in \mathbb{R}$$

This represents an indication that we can use reduce to Separable.

ex:-

$$\frac{dy}{dx} = \frac{1}{\sqrt{3y + x - 1}}$$

$\rightarrow$ u

$\rightarrow$ $$\frac{dy}{dx} = f(3y + x - 1)$$

$\rightarrow$ let $u = 3y + x - 1$

$\rightarrow$ Rewrite in term of u and x

$\rightarrow$ Derive w.r.t x :-

$$\frac{du}{dx} = 3\frac{dy}{dx} + 1$$

$\rightarrow$ Solve for dy/dx

$$dy/dx = \frac{1}{3}\frac{du}{dx} - \frac{1}{3}$$

$$\frac{1}{3}\frac{du}{dx} - \frac{1}{3} = \frac{1}{\sqrt{u}}$$

$\rightarrow$ write in seprabale

$$\frac{du}{dx} = \frac{3(1 + \sqrt{u})}{\sqrt{u}}$$

Subject: ______ (Date / / :التاريخ) ______ :الموضـــوع

$\sqrt{u}\,du - 3(1+\sqrt{u})\,dx = 0$ "Divide by $1+\sqrt{u}$

$$\frac{\sqrt{u}}{1+\sqrt{u}}\,du - 3\,dx = 0$$

$$\int \frac{\sqrt{u}}{1+\sqrt{u}}\,du - 3x = 0$$

let $w = 1+\sqrt{u} \Rightarrow \sqrt{u} = w-1$

$$dw = \frac{1}{2\sqrt{u}}\,du$$

$du = 2\sqrt{u}\,dw$

$du = \cancel{2\sqrt{u}}\ 2(w-1)\,dw$

$$\int \frac{\sqrt{u}}{1+\sqrt{u}}\,du = \int \frac{w-1}{w} \cdot 2(w-1)\,dw$$

$$= 2\int \frac{w^2-2w+1}{w}\,dw = 2\int w - 2 + \frac{1}{w}\,dw$$

⋮

As we see u we sub for the original function.

$u = 3y + x - 1$

Subject: .... (Date / / :التاريخ) .... :الموضــوع

Ex:-

$$\frac{dy}{dx} = \sin(2y - x) + \frac{1}{2}$$ "1.st order non linear"

$$u = 2y - x$$

$$\frac{du}{dx} = 2\frac{dy}{dx} - 1$$

$$\frac{dx}{dx} = \frac{1}{2}\frac{du}{dx} + \frac{1}{2}$$

$$\frac{1}{2}\frac{du}{dx} + \frac{1}{2} = \sin(u) + \frac{1}{2}$$

$$\frac{du}{dx} = 2\sin(u)$$

$$2\,dx + \frac{-1}{\sin(u)} \cdot du = 0$$

Subject: (Date / / :التاريخ) :الموضــوع

2/5/2009 Sunday

Cauchy Euler

This method is useful when we have :-
Linear - Diff with non-Constant Co-effecient.

Observe :-

$$x^2 (y)^{(2)} + xy' + 2y = 0$$

$y = x^n$, for some n.

$$y' = n x^{n-1}$$
$$y'' = n(n-1) x^{n-2}$$

$$n(n-1) x^n + n x^n + 2x^n = 0$$
$x^n (n^2 - n + n + 2) = 0$, Find n.

Cauchy - Euler :-

These D.E are solved by leting $y = x^n$ and we need to find n.

The idea here, after factorizing we end with equation that all terms have same Power (degree)

Ex :-
$$y^{(3)} + \frac{1}{x^2} y' + \frac{1}{x^3} y = 0$$

Subject: (Date / / :التاريخ) :الموضــوع

$y = x^n$

$y' = n x^{n-1}$

$y'' = n(n-1) x^{n-2}$

$y''' = n(n-1)(n-2) x^{n-3}$

it will work!

Note:

$e^{xi} = \cos(x) + i\sin(x)$

$x^{a+bi} = x^a \cdot x^{bi}$

$= x^a \left(e^{\ln(x^{bi})} \right)$

$= x^a \left(e^{bi\ln(x)} \right)$

$= x^a \left(e^{b\ln(x)i} \right)$

$= x^a \left(\cos(b\ln(x)) + i\sin(b\ln(x)) \right)$

$x^{a+bi} = x^a \left[\cos(b\ln(x)) + i\sin(b\ln(x)) \right]$

ex:- • x^{-3i}

$= \cos(-3\ln(x)) + i\sin(-3\ln(x))$

$x^{\overset{a}{2}+\overset{b}{5}i} = x^2\left(\cos(5\ln(x)) + i\sin(5\ln(x))\right)$

Subject: (Date / / :التاريخ) :الموضـــوع

Solve:-

$$x^2y^{(2)} + xy' + 4y = 0$$

Solution :-

$y = x^n$, find n

$y' = n x^{n-1}$

$y^{(2)} = n(n-1)\,x^{n-2}$

Sub in the Diff Equ

$n(n-1)\,x^n + n x^n + 4x^n = 0$

$x^n(\,n(n-1) + n + 4\,) = 0$

$x^n\,(n^2 - n + n + 4) = 0$

Char D.E :-

$n^2 + 4 = 0$

$n = \pm 2i$ (now we use only one solution to get 2-independt solution)

$x^{2i} \longrightarrow y_1 = \cos(2\,\ln(x))$

$y_2 = \sin(2\,\ln(x))$

$y_h = C_1 \cos(2\,\ln(x)) + C_2 \sin(2\,\ln(x))$

$$y^{(2)} + \frac{3}{x} y' + \frac{y}{x^2} = 0$$

Solution:-

$$y = x^n$$

$$y' = n x^{n-1}$$

$$y'' = n(n-1) x^{n-2}$$

$$(n^2 - n)(x)^{n-2} + 3n x^{n-2} + x^{n-2}$$

$$x^{n-2}(n^2 - n + 3n + 1) = 0$$

Chare D.E:-

$$n^2 + 2n + 1 = 0$$

$$(n+1)(n+1) = 0$$

$$n = -1$$

$$y_1 = x^{-1}$$

$$y_1 = x^{-1} \ln(x)$$

$$y_h = C_1 x^{-1} + C_2 x^{-1} \ln(x)$$

Subject: ______ (Date / / :التاريخ) ______ :موضوع

$$\frac{dy}{dx} = \frac{1}{x(x-y)}$$

Trick:-

do $\frac{dx}{dy}$

$$\frac{dx}{dy} = x(x-y)$$ "x dependent, y-independent"

$$\frac{dx}{dy} = x^2 - xy$$

$$\frac{dx}{dy} + \overset{Q(x)}{y}x = x^2$$

it's bernoly!

$n=2$

$u = x^{-1}$

$1-n = -1$

$$\frac{du}{dy} + (1-n)yu = (1-n)$$

$$u' \underset{Q(x)}{-y}u = \underset{F(y)}{-1}$$

$$I = e^{\int Q(y)\,dy} = e^{\int -y\,dy} = e^{-y^2/2}$$

$$u(y) = \frac{\int I . F(y)\,dy}{I} = \frac{\int e^{-\frac{1}{2}y^2} . (-1)}{e^{-\frac{1}{2}y^2}}$$ "Difficult to integrate"

Assume

$u(y) = y^2 + e^y + 3$

Subject: (Date / / :التاريخ) :الموضوع

reduce to Seprabler

$$\frac{dy}{dx} = f(ay+bx+C)$$

$\neq 0$

$$y' = \sqrt{y+x}$$

$$u = y+x$$

$$\frac{du}{dx} = \frac{dy}{dx}+1$$

$$\frac{dy}{dx} = \frac{du}{dx}-1$$

$$\frac{du}{dx}-1 = \sqrt{u}$$

$$\frac{du}{dx} = \sqrt{u}+1$$

$$du-(\sqrt{u}+1)\,dx=0$$

$$\frac{1}{\sqrt{u}+1}\,du - dx = 0$$

$$\frac{1}{\sqrt{u}+1}\,du - x = C$$

$$w = \sqrt{u}+1 \Rightarrow \sqrt{u} = w-1$$

$$dw = \frac{1}{2\sqrt{u}}\,du$$

$$du = 2\sqrt{u}\,du$$

$$du = 2(w-1)\,dw$$

$$\int \frac{1}{w}(2(w-1))\,dw$$

$$= 2\int \frac{w-1}{w}\,dw$$

$$= 2(w - \ln|w|)$$

$$w = \sqrt{u}+1$$

$$= \sqrt{x+y}+1$$

$$= 2(w - \ln|w|) - x = C$$

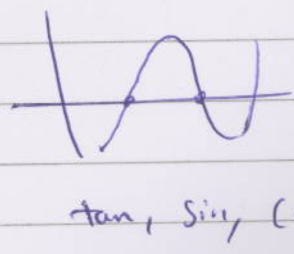
$$\cancel{2(\sqrt{x+y} - \ln|(x+y)|) - x = C}$$

$$2(\sqrt{x+y} - \ln|\sqrt{x+y}+1|) - x = C$$

(8) $x(t) = 0.6\cos(10t) + 0.5\sin(10t$

second time

$\frac{1}{2}$ Period + first time.



tan, Sin, Cos are all in Rad