3. (a) Let $S = \{a_1x + a_2x^2 + a_3x^3 \mid a_1, a_2, a_3 \in \mathbb{R}\}$.
Suppose $p(x), q(x) \in S$ and $\alpha \in \mathbb{R}$.
Then $p(x) = a_1x + a_2x^2 + a_3x^3$ and $q(x) = b_1x + b_2x^2 + b_3x^3$ for some $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$.
Now,
$$\begin{aligned} p(x) + q(x) &= (a_1x + a_2x^2 + a_3x^3) + (b_1x + b_2x^2 + b_3x^3) \\ &= (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \in S. \end{aligned}$$
Also, $\alpha p(x) = \alpha(a_1x + a_2x^2 + a_3x^3) = \alpha a_1x + \alpha a_2x^2 + \alpha a_3x^3 \in S$.
Hence S is a subspace of P_3.

(b) Let $S = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R},\ a_0 + a_1 + a_2 + a_3 = 0\}$.
Suppose $p(x), q(x) \in S$ and $\alpha \in \mathbb{R}$.
There exist $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ such that $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_0 + a_1 + a_2 + a_3 = 0$ and $q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ where $b_0 + b_1 + b_2 + b_3 = 0$.
Then $p(x) + q(x) = (a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0 + b_1x + b_2x^2 + b_3x^3)$
$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$.
Now,
$$\begin{aligned} (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) &= (a_0 + a_1 + a_2 + a_3) + (b_0 + b_1 + b_2 + b_3) \\ &= 0 + 0 = 0 \end{aligned}$$
and so $p + q \in S$.
Also $\alpha p(x) = \alpha(a_0 + a_1x + a_2x^2 + a_3x^3) = \alpha a_0 + \alpha a_1x + \alpha a_2x^2 + \alpha a_3x^3$.
Now, $\alpha a_0 + \alpha a_1 + \alpha a_2 + \alpha a_3 = \alpha(a_0 + a_1 + a_2 + a_3) = 0$ and so $\alpha p \in S$.
Hence S is a subspace of P_3.

4. (a) Let $S = \{f \in V \mid f(0) = 0\}$.
Suppose $f, g \in S$ and $\alpha \in \mathbb{R}$.
Then $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$ and so $f + g \in S$.
Also $(\alpha f)(0) = \alpha f(0) = \alpha 0 = 0$ so $\alpha f \in S$.
Hence S is a subspace of V.

(b) Let $S = \{f \in V \mid f(0) = 1\}$.
Take $f(x) = 1 + x$ and $g(x) = 1 + x^2$. Now $f(0) = g(0) = 1$ so $f, g \in S$, but $(f + g)(0) = f(0) + g(0) = 2$. Hence $f + g \notin S$ and therefore S is not a subspace of V.

(c) Let $S = \{f \in V \mid f \text{ is continuous}\}$
Suppose $f, g \in S$ and $\alpha \in \mathbb{R}$.
Then, as was discussed in the first year calculus course, $f + g$ and αf are continuous and so lie in S. Hence S is a subspace of V.

(d) Let $S = \{f \in V \mid f \text{ is differentiable}\}$.
Suppose $f, g \in S$ and $\alpha \in \mathbb{R}$.
As was shown in the the first year calculus course, $f + g$ and αf are differentiable and so lie in S. Hence S is a subspace of V.

5. (a) $0.u + 0.u = (0 + 0).u = 0.u = 0.u + \underline{0}$.
Hence, by the cancellation law (proved in the handout), $0.u = \underline{0}$.

(b) It is proved in the handout that $\alpha.\underline{0} = \underline{0}$ for all $\alpha \in \mathbb{R}$. Hence
$\alpha.(-u) + \alpha.u = \alpha.((-u) + u) = \alpha.\underline{0} = \underline{0} = -(\alpha.u) + \alpha.u$.
Hence, by the cancellation law, $\alpha.(-u) = -(\alpha.u)$.

(c) It is proved in (a) above that $0.u = \underline{0}$ for all $u \in V$. Hence
$(-\alpha).u + \alpha.u = (-\alpha + \alpha).u = 0.u = \underline{0} = -(\alpha.u) + \alpha.u$.
Hence, by the cancellation law, $(-\alpha.u) = -(\alpha.u)$.