EXAMPLES 2: VECTOR SPACES AND SUBSPACES — SOLUTIONS

1. (a) Let $S = \{(a,0,0) \in \mathbb{R}^3 \mid a \in \mathbb{R}\}$.
Suppose $u, v \in S$ and $\alpha \in \mathbb{R}$. Hence $u = (a_1,0,0)$ and $v = (a_2,0,0)$ for some $a_1, a_2 \in \mathbb{R}$.
Now $u+v = (a_1,0,0)+(a_2,0,0) = (a_1+a_2,0,0) \in S$.
Moreover $\alpha u = \alpha(a_1,0,0) = (\alpha a_1,0,0) \in S$.
Thus S is a subspace of $\mathbb{R}^3$.

(b) Let $S = \{(a,1,0) \in \mathbb{R}^3 \mid a \in \mathbb{R}\}$.
Then $(1,1,0) \in S$ but $2(1,1,0) = (2,2,0) \notin S$ and so S is not a subspace of $\mathbb{R}^3$.

(c) Let $S = \{(a,3a,2a) \in \mathbb{R}^3 \mid a \in \mathbb{R}\}$.
Suppose $u, v \in S$ and $\alpha \in \mathbb{R}$.
Then $u = (a_1,3a_1,2a_1)$ and $v = (a_2,3a_2,2a_2)$ for some $a_1, a_2 \in \mathbb{R}$.
Hence $u+v = (a_1,3a_1,2a_1)+(a_2,3a_2,2a_2) = (a_1+a_2,3a_1+3a_2,2a_1+2a_2)$
$= (a_1+a_2,3(a_1+a_2),2(a_1+a_2)) \in S$.
Morever $\alpha u = \alpha(a_1,3a_1,2a_1) = (\alpha a_1,\alpha 3a_1,\alpha 2a_1) = (\alpha a_1,3(\alpha a_1),2(\alpha a_1)) \in S$.
Hence S is a subspace of $\mathbb{R}^3$.

(d) Let $S = \{(x,y,z) \in \mathbb{R}^3 : 2x-3y+z=0\}$.
Suppose $u, v \in S$ and $\alpha \in \mathbb{R}$.
Then $u = (x_1,y_1,z_1)$ where $2x_1-3y_1+z_1=0$ and $v = (x_2,y_2,z_2)$ where $2x_2-3y_2+z_2 = 0$.
Then $u+v = (x_1+x_2,y_1+y_2,z_1+z_2)$.
But $2(x_1+x_2)-3(y_1+y_2)+(z_1+z_2) = (2x_1-3y_1+z_1)+(2x_2-3y_2+z_2) = 0+0 = 0$ and so $u+v \in S$.
Also $\alpha u = (\alpha x_1,\alpha y_1,\alpha z_1)$ and $2(\alpha x_1)-3(\alpha y_1)+\alpha z_1 = \alpha(2x_1-3y_1+z_1) = \alpha.0 = 0$.
Thus $\alpha u \in S$.
Hence, by the Subspace Test, S is a subspace of $\mathbb{R}^3$.

(e) Let $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1, x,y,z \in \mathbb{R}\}$.
Take $u = (1,0,0)$ and $v = (0,1,0)$. Now $u, v \in S$
but $u+v = (1,0,0)+(0,1,0) = (1,1,0)$ and $1^2+1^2+0^2 = 2$ so $u+v \notin S$.
Hence S is not a subspace of $\mathbb{R}^3$.

2. (a) Let $S = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{22} \mid a,b,c,d \in \mathbb{Z}\}$.
Then $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in S$ but $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \notin S$.
Hence S is not a subspace of M_{22}.

(b) Let $S = \{A \in M_{22} \mid A = A^t\}$.
Suppose $A, B \in S$ and $\alpha \in \mathbb{R}$.
Then $(A+B)^t = A^t+B^t = A+B$ and so $A+B \in S$.
Also $(\alpha A)^t = \alpha A^t = \alpha A$, so $\alpha A \in S$.
Hence S is a subspace of M_{22}.

(c) Let $S = \{A \in M_{22} \mid \det A = 0\}$.
Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $A+B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Now $\det A = \det B = 0$, but $\det(A+B) = 1$.
Thus $A, B \in S$, but $A+B \notin S$.
Hence S is not a subspace of M_{22}.