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$\boxed{10}$

MTH221, LINEAR ALGEBRA, SECOND EXAM FALL 2006

Question 1. *Let* $D = Span\{\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 6 \\ -12 \\ 18 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -10 \end{bmatrix}\}$

1) Find a basis for D.

$$\begin{bmatrix} 2 & -4 & 6 \\ -2 & 5 & -6 \\ 6 & -12 & 18 \\ -2 & 4 & -10 \end{bmatrix}$$

$\frac{1}{2}R_1$

$2R_1 + R_2 \to R_2$

$2R_1 + R_4 \to R_4$

$-6R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

basis for $D = \left\{ \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -10 \end{bmatrix} \right\}$

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2) Is $D = R^3$? Explain

yes $D = R^3$

because D contains three independent elements which are $\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -10 \end{bmatrix}$.

span $\left\{ \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -10 \end{bmatrix} \right\} = R^3$

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Question 2. *1) Is Span*$\{-5+6x+2x^3, -10+10x-5x^3, 30+60x-5x^2+2x^3\} = P_4$*? Explain*

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no because span $\{-5+6x+2x^3, -10+10x-5x^3, 30+60x-5x^2+2x^3\}$

doesn't contain four elements, so

we can't find four independent

elements in the span so it is

not equall P_4

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2) Is Span$\{1, x\} =$ *Span*$\{10+x, 20+4x\}$*? Explain*

$a_0 + a_1 x$

$\because A = \text{span}\{1, x\}$

$B = \text{span}\{10+x, 20+4x\}$

$\because A \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow$ 2 independent elements so $A = R^2$

$\because B \rightarrow \begin{bmatrix} 10 & 1 \\ 20 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{10} \\ 0 & 2 \end{bmatrix}$ 2elements are independent $B = R^2$

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So yes span $\{1, x\} =$ span $\{10+x, 20+4x\} = R^2$

3) Let $D = \left\{ \begin{bmatrix} a \\ 3a+1 \\ b \end{bmatrix} \mid a, b \in R \right\}$*. Is D a subspace of* R^3*? If yes, then find a basis for D.*

1) $\because v_1 \in D$ so $v_1 = \begin{bmatrix} m \\ 3m+1 \\ n \end{bmatrix} \in R^3$

$v_2 \in D$ so $v_2 = \begin{bmatrix} L \\ 3L+1 \\ k \end{bmatrix} \in R^3$

$\because v_1 + v_2 = \begin{bmatrix} m+L \\ 3(m+L)+2 \\ k+n \end{bmatrix} \notin D$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in R^3$ but $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin D$

So D is not a sub space

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Question 3. *1)Find a basis for* R^4 *that contains the following two independent* ~~*matrices*~~ elements: $\begin{bmatrix} 2 \\ 6 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} -4 \\ -12 \\ 9 \\ -14 \end{bmatrix}$.

$$\begin{array}{cccc} 4 & 12 & -8 & 16 \end{array}$$

$$\begin{bmatrix} 2 & 6 & -4 & 8 \\ -4 & -12 & 9 & -14 \end{bmatrix} \quad 2R_1+R_2 \to R_2$$

$$\begin{bmatrix} 2 & 3 & -2 & 4 \\ 0 & 7 & 4 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

so I need to enter two independent elements

So basis for R^4 = $\left\{ \begin{bmatrix} 2 \\ 6 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} -4 \\ -12 \\ 9 \\ -14 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right\}$

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a_4

SOLVE ONLY ONE OF THE FOLLOWING TWO QUESTIONS:

2) Let A *be a* 3×5 *matrix and* a_4 *be the fourth column of* A*. Find a solution to the system* $AX = a_4$*. How many solutions does the system* $AX = a_4$ *have? EXPLAIN*

3) Given that v_1, v_2 *are independent elements in a vector space* V*, and* $v_3 \notin Span\{v_1, v_2\}$*. Show that* v_1, v_2, v_3 *are independent elements of* V*.*

~~∵ v_1, v_2 are independent~~

~~so $0 = \alpha_1 v_1 + \alpha_2 v_2$, $\alpha_1 = \alpha_2 = 0$~~

~~∵ $v_3 \notin$ span $\{v_1, v_2\}$~~

~~so $v_3 \neq a v_1 + b v_2$, $a, b \in R$~~

AM 1

∵ $\underset{3\times5}{A}\ \underset{5\times1}{X} = \underset{3\times1}{a_4}$ ∵ $a_4 \in col(A)$ so $AX = a_4$ have a solution.

∵ dim(N(A)) + colum(A) = n. colomns

2 + 3 = 5

so there are many solutions.

2)

$$A = \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_4 \end{array} \\ \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{34} \end{array}\right] \end{array}$$

$$\underset{3\times 5}{A}\ \underset{5\times 1}{x} = \underset{3\times 1}{a_4}$$

$\because a_4 \in \text{col } A$ so $a_4 \in \text{span}\{a_1, a_2, a_3, a_4, a_5\}$

$$a_4 = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4 + \alpha_5 a_5$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So

$$a_{14} = x_1 a_{11} + x_2 a_{12} + x_3 a_{13} + x_4 a_{14} + x_5 a_{15}$$

so $x_4 = 1$ and $x_1 a_{11} + x_2 a_{12} + x_3 a_{13} + x_5 a_{15} = 0$

$x_1 a_{21} + x_2 a_{22} + x_3 a_{23} + x_5 a_{25} = 0$

$x_1 a_{31} + x_2 a_{32} + x_3 a_{33} + x_5 a_{35} = 0$

So $x_4 = 1$ $\quad x_1, x_2, x_3, x_5 \in \mathbb{R}$, x_1, x_2, x_3, x_5 are free variables

There are many solutions.

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Question 4. *1)Let* $F = \{f(x) \in P_3 \mid \int_0^1 f'(x)dx = 0\}$.
a)Show that F is a subspace of P_3.

$\because$ $f(x) = a_1 + a x + a_2 x^2$

$f'(x) = a_1 + 2 a_2 x$

$m(x) \in F$ so $\int_0^1 m'(x)\,dx = 0$

$L(x) \in F$ so $\int_0^1 L'(x)\,dx = 0$

$\because$ $m(x) + L(x) = \int_0^1 m'(x)\,dx + \int_0^1 L'(x)\,dx = 0 + 0$

$= \int_0^1 m'(x) + L'(x)\,dx = 0$

so $m(x) + L(x) \in F$

$\because$ $\alpha \in R$ $m(x) \in F$

$\because$ $\alpha \int_0^1 m'(x)\,dx = \int_0^1 \alpha\, m'(x)\,dx = 0$

so $\alpha\, m(x) \in F$

so F is a subspase of P_3

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b)Find a basis for F.

$\because$ $\int_0^1 a_1 + 2 a_2 x \, dx = 0$

$a_1 x + a_2 x^2 \Big|_0^1 = a_1 x + a_2 x^2 = 0$

so $a_1 + a_2 = 0 \longrightarrow a_1 = -a_2$

so $a_2, a_0 \in R$
free variables

$\because$

a_0	a_2	a_1		
1	0	$\rightarrow$ 0	$\longrightarrow$	1
0	1	$\rightarrow$ -1	$\longrightarrow$	$x^2 - x$

so basis for $F = \{ 1, -x + x^2 \}$

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Question 5. *1) Given that* A *is a* 3×4 *matrix and* A *is row equivalent to* $\begin{bmatrix} 3 & 6 & -2 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *If* $a_1 = \begin{bmatrix} 5 \\ 2 \\ -4 \end{bmatrix}$. *Find* A.

$\therefore$ basis for Row (A) = $\left\{ \begin{bmatrix} 3 \\ 6 \\ -2 \\ 9 \end{bmatrix} \right\}$

basis for col (A) = $\left\{ \begin{bmatrix} 5 \\ 2 \\ -4 \end{bmatrix} \right\}$

$$A = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 5 & 10 & \frac{-10}{3} & 15 \\ 2 & 4 & \frac{-4}{3} & 6 \\ -4 & -8 & \frac{8}{3} & -12 \end{bmatrix} \end{array}$$

$a \begin{bmatrix} 3 \\ 6 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$\therefore 3a = 5 \longrightarrow a = \frac{5}{3}$

$b_1 = 6 \times \frac{5}{3} = 10 \quad b_2 = -2 \times \frac{5}{3} = \frac{-10}{3} \quad b_3 = 9 \times \frac{5}{3} = 15$

$a \begin{bmatrix} 3 \\ 6 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$a = \frac{2}{3} \longrightarrow b_1 = 6 \times \frac{2}{3} = 4 \quad b_2 = -2 \times \frac{2}{3} = \frac{-4}{3} \quad b_3 = 9 \times \frac{2}{3} = 6$

$a \begin{bmatrix} 3 \\ 6 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} -4 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \longrightarrow a = \frac{-4}{3} \longrightarrow b_1 = 6 \times \frac{-4}{3} = -8, \quad b_2 = -2 \times \frac{-4}{3} = \frac{8}{3}, \quad b_3 = 9 \times \frac{-4}{3} = -12$

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2)Let $T : R^2 \longrightarrow R$ *be a linear transformation such that* $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -10$, *and* $\begin{bmatrix} 4 \\ 10 \end{bmatrix} \in Ker(T)$. *Find* $T\left(\begin{bmatrix} 20 \\ 53 \end{bmatrix}\right)$

$T \begin{bmatrix} 4 \\ 10 \end{bmatrix} = 0$

$\therefore a \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ 53 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \left[\begin{array}{cc|c} 0 & 4 & 20 \\ 1 & 10 & 53 \end{array}\right] \end{array}$

$\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \left[\begin{array}{cc|c} 1 & 10 & 53 \\ 0 & 4 & 20 \end{array}\right] \end{array} \longrightarrow \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \left[\begin{array}{cc|c} 1 & 10 & 53 \\ 0 & 1 & 5 \end{array}\right] \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right]$

$a = 3 \quad b = 5$

so $T\left(\begin{bmatrix} 20 \\ 53 \end{bmatrix}\right) = 3\, T \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 5\, T \begin{bmatrix} 4 \\ 10 \end{bmatrix} = 3 \times \cancel{10} = -30$

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$3\times 1 = 3\times 4 \quad 4\times 1$

Question 6. *Let* $T : R^4 \longrightarrow R^3$ *be a linear transformation from* R^4 *into* R^3 *such that* $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -2x_1 + 2x_2 - 3x_3 \\ 3x_1 - 3x_2 + 6x_3 \end{bmatrix}$

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1)Find the standard matrix representation of T.

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -2 & 2 & -3 & 0 \\ 3 & -3 & 6 & 0 \end{bmatrix}$$

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2) Find a basis for Ker(T).

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 0 \\ -2 & 2 & -3 & 0 & 0 \\ 3 & -3 & 6 & 0 & 0 \end{array}\right] \xrightarrow[\substack{-3R_1 + R_3 \to R_3 \\ -2R_2 + R_1 \to R_1}]{2R_1 + R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x_1 &= x_2 \\ x_3 &= 0 \end{aligned} \quad , x_2, x_4 \in R$$

$$N(A) = \left\{ \begin{bmatrix} x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} \;\middle|\; x_2, x_4 \in R \right\}$$

so basis for Ker(T) =

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3) Find a basis for the RANGE of T.

basis for Range of T

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} \right\}$$