PROBLEM SET 8

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Exercise 1. *Consider the group* $(\mathbb{Z}, +)$*, and let* H *be a subgroup of* $\mathbb{Z}$*. Show that* $H = n\mathbb{Z}$ *for some* $n \geq 1$*.*

Proof. We know $\mathbb{Z}$ is cyclic and generated by 1. Now let $a \in H$, then $a = 1^m$ for some $m \in \mathbb{Z}$. Thus each element of H is a power of 1. Let $k = \min\{m \in \mathbb{Z}^+ \mid 1^m \in H\}$. Then $H = \langle 1^k \rangle = \langle k \rangle$, this choice of k follows from the proof in class. So we have $H = \langle k \rangle = \{\ldots, -2k, -k, 0, k, 2k, \ldots\} = k \times \{\ldots, -2, -1, 0, 1, 2, \ldots\} = k\mathbb{Z}$. □

Exercise 2 (a). *Let* $M = \{2, 4, 6, 8, 10, 12\}$*. Show that* $(M, \times_{14})$ *is a cyclic group.*

Solution. We note that 8 is the identity of this group, and this verifiable from the following multiplicative table.

$\times_{14}$	2	4	6	8	10	12
2	4	8	12	2	6	10
4	8	2	10	4	12	6
6	12	10	8	6	4	2
8	2	4	6	8	10	12
10	6	12	4	10	2	8
12	10	6	2	12	8	4

An element of order 6 in the group M is 10, thus M is cyclic. □

Exercise 2 (b). *Let* $H = \{6, 8\}$*. Show that* $(H, \times_{14})$ *is a normal subgroup of* M *(*M *as in part (a)).*

Solution. H is clearly a subgroup of M since 6 is an element of order 2, and $e = 8 \in H$. Normality of H is clear since M is an abelian group. □

Exercise 2 (c). *Now consider the quotient group $(M/H, \wedge)$. Find all elements of M/H. Find the order of each element of M/H. Is $(M/H, \wedge)$ a cyclic group? Explain.*

Solution. Finding the elements of M/H amounts to finding the left cosets of H. We first note that $[M : H] = 3$. Computing the left cosets of H, we get $M/H = \{H, 2*H, 4*H\} = \{\{6, 8\}, \{12, 2\}, \{10, 4\}\}$. Also, $|H| = 1$, $|2 * H| = 3$, and $|3 * H| = 3$, and this is clear since M/H is of prime order. This also implies that M/H is cyclic. □

Exercise 3 (a). *Let $(M, *)$ be a finite abelian group of order n. Let $a_1, a_2, \ldots, a_n$ be the elements of M. If n is an odd number, show that $a_1 * a_2 * a_3 * \ldots * a_n = e$*

Proof. Take all a_i's $\neq e$. Each a_i must have a distinct inverse (that is, there is no element of order 2 since the order of the group is odd). Since M is abelian, we can write the elements in the following form: $(a_i * a_i^{-1}) * (a_j * a_j^{-1}) * \ldots * (a_n * a_n^{-1}) = e$ and we are done. □

Exercise 3 (b). *If n is an even number, show that $x = a_1 * a_2 * a_3 * \ldots * a_n \in M$ and x is of order 2.*

Proof. Let $\lambda = \{m \in M \mid |m| \neq 2\}$. For every element a of order greater than 2, there exists a distinct element a^{-1} such that $a * a^{-1} = a^{-1} * a = e$. There is an even number of such elements since we have some arbitrary number of pairs of a, a^{-1}. Thus we can write elements of order greater than 2 in the following form: $(a_i * a_i^{-1}) * (a_j * a_j^{-1}) * \ldots * (a_n * a_n^{-1}) = e$. Note, however, than the order of λ is odd since the inverse of the identity is trivially itself. Thus to show (b), we only consider the product of elements of order 2. That is, elements of the set $M \setminus \lambda$. Clearly it is nonempty since $|M| > |\lambda|$, and $|M|$ and $|\lambda|$ are even and odd, respectively, so their difference is odd. If $M \setminus \lambda$ is a singleton, then we are done. Otherwise assume $|M \setminus \lambda| > 1$, and take distinct $a, b \in M \setminus \lambda$. Clearly $a * b \neq e$. We have $|a * b| = 2$ since $(a * b)^2 = a^2 * b^2 = e$. Thus the product of all elements of M must give us an element of order 2. □

Exercise 3 (c). *(From the book) Consider the numbers $1, 2, 3, \ldots, 10$: can you add $+$ or $-$ in front of each number so that when you add them, then you get zero?*

Solution. No, you cannot. The sum of the numbers $1, 2, 3, \ldots, 10$ can easily be found to be 55. Notice that adding a minus in front of some digit k constitutes to subtracting $2k$ from the sum. Since the sum 55 is odd, subtracting distinct even numbers will never yield 0. □

Exercise 3 (d). *(From the book) consider the numbers* $1, 2, 3, \ldots, 19$*: can you add* $+$ *or* $-$ *in front of each number so that when you add them, then you get zero?*

Solution. Yes. In fact, $(1+2+3+4+5+6+7+8+9+(-10)+11+12+13+14+(-15)+(-16)+(-17)+(-18)+(-19)) = 0$. □