PROBLEM SET 2

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Exercise 1. *Let* $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 1 & 7 & 2 \end{pmatrix}$. *Write* α *as a composition of disjoint cycles.*

Solution. $\alpha = (1\ 3\ 5) \circ (2\ 4\ 6\ 7)$. □

Exercise 2. *Let* X *be an infinite set, and* $f \in X^X$. *We say* f *is almost identical iff the set* $\{\alpha \in X \mid f(\alpha) \neq \alpha\}$ *is finite. Now let* $M = \{f \in X^X \mid \text{f is almost identical}\}$. *Show that* $(M, \circ)$ *is a monoid.*

Proof. Associativity follows trivially, for $\circ$ is an associative binary operation on X^X, and M is a subset of X^X. There exists $e(\alpha) = \alpha$ in M since $\{\alpha \in X \mid e(\alpha) \neq \alpha\} = \emptyset$ is finite. To show closure, take $f, g \in M$. Then we know that $F = \{\alpha \in X \mid f(\alpha) \neq \alpha\}$ and $G = \{\alpha \in X \mid g(\alpha) \neq \alpha\}$ are finite. We have to show that $H = \{\alpha \in X \mid f \circ g(\alpha) \neq \alpha\}$ is finite. We denote the set $X - G$ to be $\{\alpha \in X \mid \alpha \notin G\}$. Since $g(\alpha) = \alpha$ for all but finitely many $\alpha \in X$ (to put it more precisely, $g(\alpha) = \alpha$ for all $\alpha \in X - G$), we have $J = \{\alpha \in X - G \mid f \circ g(\alpha) \neq \alpha\} = \{\alpha \in X - G \mid f(\alpha) \neq \alpha\}$ is finite since J is a subset of F. It remains to show that $E = \{\alpha \in G \mid f \circ g(\alpha) \neq \alpha\}$ is finite, but G is finite and thus E must also be finite. Thus H is finite, and $f \circ g$ is almost identical. We have thus shown that $(M, \circ)$ is a monoid. □

Exercise 3. *Let* $n \geq 3$. *Show that* S_n *is a nonabelian group.*

Proof. We know that S_n is a group. Take two 2-cycle elements $(1\ 2)$, $(2\ 3)$, denoted as α and β, respectively. Then $\alpha \circ \beta = (1\ 2) \circ (2\ 3) = (1\ 2\ 3)$, whereas $\beta \circ \alpha = (2\ 3) \circ (1\ 2) = (1\ 3\ 2)$. Note that $\alpha, \beta \in S_n$ for all $n \geq 3$, but we have shown that α and β do not commute. Thus the conclusion follows. □

Exercise 4. *Let α, β be disjoint cycles, and n, m be positive integers.*

(a) *Show that* $\alpha^m \circ \beta^n = \beta^n \circ \alpha^m$

(b) *Show that* $\alpha^{-1} \circ \beta = \beta \circ \alpha^{-1}$ *and* $\alpha^{-1} \circ \beta^{-1} = \beta^{-1} \circ \alpha^{-1}$

Proof. For (a), let $\alpha = (a_1\ a_2\ \cdots\ a_j)$, and $\beta = (b_1\ b_2\ \cdots\ b_k)$. Since α and β are disjoint, $a_r \neq b_s$ for all $r \in \{1, 2, \ldots, j\}$ and $s \in \{1, 2, \ldots, k\}$. Now assume for the sake of contradiction that $\alpha^m \circ \beta^n \neq \beta^n \circ \alpha^m$. That is, α^m and β^n are not disjoint. Then we can find an entry k that belongs to both α^m and β^n. Consider α^m first. Since k belongs to α^m, then by Lemma 1, k will equal to a_y for some $y \in \{1, 2, \ldots, j\}$. That is, k is an entry in α. But the same argument can be used to show that k is an entry in β. We have just shown α and β are not disjoint, contradiction. Thus our original assumption does not hold, and α^m and β^n are disjoint. The conclusion follows since disjoint implies commutativity.

For (b), $\alpha^{-1} = (a_j\ \cdots\ a_2\ a_1)$ and $\beta^{-1} = (b_k\ \cdots\ b_2\ b_1)$, so α^{-1} and β^{-1} consist of all entries of α and β, respectively. But we know that α and β share no common entries, so α^{-1} and β are disjoint, and thus commute. By the same token, α^{-1} and β^{-1} are disjoint, and so they also commute. $\square$

Lemma 1. *For all positive integers m, α^m is written as the composition of disjoint cycles with entries from α.*

Proof 1. We induct on m. The base case is trivial. Now suppose inductively that α^m is written as the composition of disjoint cycles with entries from α. Then we have to show that α^{m+1} is written as the composition of disjoint cycles with entries from α. That is, $\alpha^m \circ \alpha$ is written as the composition of disjoint cycles with entries from α. But by the inductive hypothesis, α^m consists of entries from α, and hence α^{m+1} consists of entries of α, and is written as the composition of disjoint cycles with entries from α. Thus the induction is closed. $\square$

Proof 2. Take $a_1 \in \alpha$, then the permutation α^m will map a_1 into a_{1+m}. Similarly, a_{1+m} will be mapped into a_{1+2m}, and so on, until for some $x \in \mathbb{N}$, $a_{1+xm} = a_1$. If we have exhausted the entries in α, then we are done. Otherwise, find the next element not in the first cycle, and repeat the process, so that α^m will be written as a composition of disjoint cycles with entries from α. $\square$