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MTH512 Homework 3 — 9 April 2023 — Advanced Linear Algebra

1. Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

   (i) Find $A \otimes B$

   Answer:

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$$A \otimes B = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 4 & 2 & 0 & 4 & 2 \\ 0 & 0 & -4 & 0 & 0 & -4 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 2 & 1 & 0 & 4 & 2 \\ 0 & 0 & -2 & 0 & 0 & -4 \end{bmatrix}$$

   (ii) Find $|A \otimes B|$

   Answer: We know that $|A \otimes B| = |A|^m |B|^n$, where m is the size of B and n is the size of A.

$$|A| = 2 \times 2 - 1 \times 2 = 4 - 2 = 2$$

$$|B| = -2 \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2(2 - 0) = -4$$

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   Therefore,

$$|A \otimes B| = (2)^3 (4)^2 = 128$$

   (iii) Find $Trace(A \otimes B)$

   Answer: We know that $Trace(A \otimes B) = Trace(A) \times Trace(B)$.

$$Trace(A) = 2 + 2 = 4$$

$$Trace(B) = 1 + 2 - 2 = 1$$

   Therefore,

$$Trace(A \otimes B) = Trace(A) \times Trace(B) = 4 \times 1 = 4$$

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2. (a) Let $T : V \to V$, $L : W \to W$ be $\mathbb{R}$-homomorphisms such that α is an eigenvalue of T, and β is an eigenvalue L. Prove that $\alpha\beta$ is an eigenvalue of $T \otimes L$. [hint: note that $T \otimes L(v \otimes w) = T(v) \otimes L(w)$]

Answer: Let $T : V \to V$ be a linear transformation such that α is an eigenvalue of T $\Rightarrow$ $v \in V$ such that $T(v) = \alpha v$. Similarly, Let $L : W \to W$ be a linear transformation such that β is an eigenvalue of T $\Rightarrow$ $w \in W$ such that $T(w) = \beta w$. Take $T \otimes L(v \otimes w) = T(v) \otimes L(w) = \alpha v \otimes \beta w = \alpha\beta(v \otimes w) \Rightarrow \alpha\beta$ is an eigenvalue of $T \otimes L$.

$\boxed{\text{for some v not = 0\_V}}$

$\boxed{\text{for some w not = 0\_w}}$

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$, $L : \mathbb{R}^3 \to \mathbb{R}^3$ be R-homomorphisms such that $-3, -3, -7$ are the eigenvalues of T, and $4, 1, 2, 3$ are the eigenvalues of L. If $E_{-3}(T) = span\{(1, 0, 1), (-1, 1, -1)\}$ and $E_4(L) = span\{(1, 0, 1, 0)\}$. In view of (a), find $E_{-12}(T \otimes L)$ I claim if $-3, -3, -6$ are the eigenvalues of T, then I cannot find $E_{-12}(T \otimes L)$ unless more information is provided, why?

Answer: If v is an eigenvector of T, and w is an eigenvector of L, then $v \otimes w$ is an eigenvector of $T \otimes L$. Therefore,

$$v_1 = (1, 0, 1) \otimes (1, 0, 1, 0) \sim (1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0)$$

and

$$v_2 = (-1, 1, -1) \otimes (1, 0, 1, 0) \sim (-1, 0, -1, 0, 1, 0, 1, 0, -1, 0, -1, 0)$$

are the eigenvectors of $T \otimes L$ corresponding to -12. By staring, these two vectors are independent. Therefore, $E_{-12} = span\{v_1, v_2\}$. If the eigenvalues of T were $-3, -3, -6$, then we have $-3 \times 4 = -12$ and $2 \times -6 = -12$. Therefore, we would need to know $E_2(L)$ and $E_{-6}(T)$ to know $E\_12(T \otimes L)$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ $L : \mathbb{R}^2 \to \mathbb{R}^2$ be $\mathbb{R}$-homomorphisms such that $T(a, b) = (2a+4b, -a-2b)$ and $L(a, b) = (-a+b, a-b)$. Find a basis for $Range(T \otimes L)$ and a basis for $Ker(T \otimes L)$. [Hint: let $\{f_1, f_2, f_3, f_4\}$ be the standard basis of $\mathbb{R}^4$, define the colinear $H : \mathbb{R}^4 \to \mathbb{R}^4$ with the standard matrix presentation $M = [H(f_1), H(f_2), H(f_3), H(f_4)]$, where $H(f_1) = T(e_1) \otimes L(e_1)$, $H(f_2) = T(e_1) \otimes L(e_2)$, ..., $H(f_4) = T(e_2) \otimes L(e_2)$. Now find the range of H and $Ker(H)$. Assume $Range(H) = span\{(3, 7, 2, 1)\}$. Then $Range(T \otimes L) = span\{3e_1 \otimes e_2 + 7e_1 \otimes e_2 + 2e_2 \otimes e_1 + e_2 \otimes e_2\}$].

Answer: We have

$$H(f_1) = T(e_1) \otimes L(e_1) = (2, -1) \otimes (-1, 1) = (-2, 2, 1, -1)$$

$$H(f_2) = T(e_1) \otimes L(e_2) = (2,-1) \otimes (1,-1) = (2,-2,-1,1)$$
$$H(f_1) = T(e_1) \otimes L(e_1) = (4,-2) \otimes (-1,1) = (-4,4,2,-2)$$
$$H(f_1) = T(e_1) \otimes L(e_1) = (4,-2) \otimes (1,-1) = (4,-4,-2,2)$$

Therefore,

$$M_H = \begin{bmatrix} -2 & 2 & -4 & 4 \\ 2 & -2 & 4 & -4 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & -2 & 2 \end{bmatrix}$$

To find $Ker(H)$, we need to find the nullspace of M_H. Using Kill below, we get

$$M_H = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Hence, $Ker(H) = span\{(1,1,0,0),(-2,0,1,0),(2,0,0,1)\}$. Therefore, $Ker(T \otimes L) = \{e_1 \otimes e_1 + e_1 \otimes e_2, -2e_1 \otimes e_1 + e_2 \otimes e_1, 2e_1 \otimes e_1 + e_2 \otimes e_2\}$. We know that $dim(Range(H)) = dim(\mathbb{R}^4) - dim(Ker(T)) = 4 - 3 = 1$. Looking at the reduced matrix, we find that the first column of the original matrix spans the whole column space. Thus, $Range(H) = span\{(-2,2,1,-1)\}$. Therefore,

$$Range(T \otimes L) = \{-2e_1 \otimes e_1 + 2e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2\}$$

3. Let D be a subspace of a real inner product space V and assume $2 \leq dim(V) = n$. Then the orthogonal space of D is denoted by $D^\perp$ and $D^\perp = \{w \in V | < w, d >= 0 \ \forall \ d \in D\}$. It is clear that if D

   has a basis $B = \{v_1, ..., v_k\}$, then $f \in D^\perp$ if and only if $< f, v_i >= 0$ for every $1 \leq i \leq k$.

   (i) Prove that $D \cap D^\perp = \{0_V\}$ [hint: Trivial, it should not exceed 2 lines]
   Answer: Suppose there is a nonzero vector f in $D \cap D^\perp \Rightarrow < f, d >= 0$ for every $d \in D$ but since $f \in D \Rightarrow f = 0_V$, Contradiction. 5

   (ii) Prove that $D + D^\perp = V$ and $dim(D + D^\perp) = dim(D) + dim(D^\perp)$ (recall that $H + F = \{h + f | h \in H, f \in F\}$)[hint: One way, assume $B = \{v_1, ..., v_k\}$ is an ORTHOGONAL basis for D. Then B

can be extended to a basis for V, how? choose v_{k+1} not in D, then $v_{k+1}, v_1, v_2, ..., v_k$ are independent, continue this process, choose v_{k+2} not in $span\{v_1, ..., v_k, v_{k+1}\}$, then $v_{k+2}, v_{k+1}, v_1, ..., v_k$ are independent. By continuing this process, we get a basis $K = \{v_1, ..., v_k, v_{k+1}, ..., v_n\}$ for V . Now for each $k+1 \leq i \leq n$, use the (modified) Gram-Schmidt process(i.e., just use the orthogonal basis of D, see(b) in Question 4) and form the elements $w_{k+1}, ..., w_n$, where $wi = vi - (\Sigma_{j=1}^{k} \frac{<v_i,v_j>}{|v_j|^2} v_j)$. Note that each w_i is orthogonal to v_j for every $1 \leq j \leq k$. Then $W = \{v_1, .., v_k, w_{k+1}, ..., w_n\}$ form a basis for V (note that since K is a basis for V, by construction, W is a basis for V !!). By construction, $D^{\perp} = span\{w_{k+1}, ..., w_n\}$. Since $D \cap D^{\perp} = 0_v$, it is clear $dim(D + D^{\perp}) = dim(D) + dim(D^{\perp}) = n$. Thus $V = D + D^{\perp}$]

Answer: We know that $D + D^{\perp} \subseteq V$. We show that $V \subseteq D + D^{\perp}$. Let $v \in V$. Let $B = \{v_1, ..., v_k\}$ be an orthogonal basis for D. Define $x = \Sigma_{i=1}^{k} \frac{<v,v_i>}{|v_i|^2} v_i$. Let $w = v - x \Rightarrow v = w + x$. We know that $x \in D$. We show that $w \in D^{\perp}$. Therefore, we need to show that $< w, v_j >= 0$ for every $v_j \in B$. We have

$$< w, v_j >=< v, v_j > - < x, v_j >$$

$$=< v, v_j > -\Sigma_{i=1}^{k} \frac{< v, v_i >}{|v_i|^2} < v_i, v_j >$$

$$=< v, v_j > - \frac{< v, v_j >}{|v_j^2|} < v_j, v_j >$$

$$=< v, v_j > - < v, v_j >= 0$$

Hence, $w \in D^{\perp} \Rightarrow v \in d + d^{\perp} \Rightarrow D + D^{\perp} = V$. Since $D \cap D^{\perp} = \{0_v\}$, $dim(D + D^{\perp}) = dim(D) + dim(D^{\perp}) = n$.

(iii) We know that $\int_0^1 f(x)\, dx$ is an inner product on P_4. Let $D = span\{x^2, 1\}$. use the hint in (i) and find $D^{\perp}$.

Answer: $f \in D^{\perp}$ if and only if $< f, v_i >= 0\ \forall v_i$ in a the basis of D. We have

$$< f, x^2 >= \int_0^1 x^2(ax^3 + bx^2 + cx + d)\, dx = \frac{a}{6} + \frac{b}{5} + \frac{c}{4} + \frac{d}{3} = 0$$

$$< f, 1 >= \int_0^1 (ax^3 + bx^2 + cx + d)\, dx = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0$$

We obtain the system

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using kill below, we get

$$\begin{bmatrix} 1 & 0 & -3 & -16 \\ 0 & 1 & \frac{15}{4} & 15 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We obtain a basis for the nullspace of the matrix $\{(3, -\frac{15}{4}, 1, 0), (16, -15, 0, 1)\}$ Therefore, a basis for $D^{\perp}$ is

$$K = \{3x^3 - \frac{15}{4}x^2 + x, 16x^3 - 15x^2 + 1\}$$

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No linear combination of the two polynomial in the basis gives a polynomial of the form $ax^2 + b$. Therefore, $D \cap D^{\perp} = \{0_{P^4}\}$

(iv) (nice, no need for Gram schmidt Thm) Let

$$D = span\{Q_1 = (1, 0, 0, 1, 0), Q_2 = (0, 1, 1, 1, 0)\}$$

. It is clear $dim(D) = 2$. Use the normal dot product on R5 and find $D^{\perp}$.[Hint: Matrix multiplication is a dot product of 2 vectors!. So construct the matrix M, 2×5, i.e., Q_1 = first row, $Q2$ = second row. Then find $Null(M)$, i.e., the solution to the homogeneous system $MX = 0$, then $D^{\perp} = span\{$ basis of Null(M)$\}$]

Answer: Let $M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$. Find $Null(M)$. We find

$$\{(0, -1, 1, 0, 0), (-1, -1, 0, 1, 0), (0, 0, 0, 0, 1)\}$$

is a basis for $Null(M) = D^{\perp}$. Therefore,

$$D^{\perp} = span\{(0, -1, 1, 0, 0), (-1, -1, 0, 1, 0), (0, 0, 0, 0, 1)\}$$

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(v) Use the "mimic dot product" on P_5. Let $D = span\{x^4 + x, x^3 + x^2 + x\}$. Find $D^{\perp}$. [hint: Translate to R5, use the normal dot product on $\mathbb{R}^5$. Stare at (iv), translate the answer of (iv) to P_5. Done]

Answer: By translation and (iv), we have

$$D^{\perp} = span\{-x^3 + x^2, -x^4 - x^3, 1\}$$

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4. (a) Let V be a normed vector space over $\mathbb{R}$. Prove that $\frac{||x+y||+||x-y||}{2} \leq ||x|| + ||y||$

Answer: We have

$$||x + y|| + ||x - y|| = ||x + y - y + y|| + ||x - y + x - x||$$

$$\leq ||x + y - y|| + ||y|| + ||x - y - x|| + ||x|| = 2(||x|| + ||y||)$$

$$\Rightarrow \frac{||x + y|| + ||x - y||}{2} \leq ||x|| + ||y||$$

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(b) Let V be a real inner product vector space and $v_1, ..., v_k$ be nonzero pairwise othogonal vectors (points) in V . Choose $Q \notin span\{v_1, ..., v_k\}$ and let $h = Q - (\Sigma_{j=1}^{k} \frac{<Q,v_j>}{|v_j|^2} v_j)$. Prove that h is orthogonal to every v_i,$1 \leq i \leq k$. [Hint: Routine calculations using the definition of inner product]

Answer: We have

$$< h, v_i >=< Q - (\Sigma_{j=1}^{k} \frac{< Q, v_j >}{|v_j|^2} v_j), v_i >$$

$$=< Q, v_i > - < \Sigma_{j=1}^{k} \frac{< Q, v_j >}{|v_j|^2} v_j, v_i >$$

$$=< Q, v_i > -\Sigma_{j=1}^{k} \frac{< Q, v_j >}{|v_j|^2} < v_j, v_i >$$

$$=< Q, v_i > -\frac{< Q, v_i >}{|v_i|^2} < v_i, v_i >$$

$$=< Q, v_i > - < Q, v_i >= 0$$

Therefore, h is orthogonal to every v_i in the list.

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(c) V be a real inner product vector space, and $D = span\{v_1, ..., v_k\}$, such that $dim(D) = k < dim(V)$. Choose a point Q in $V - D$. Then the distance between Q and D is denoted by $d(Q, D) = min\{|Q-d||d \in D\}$. Question? how do we find $d \in D$ such that $|Q-d|$ is minimum. Answer: (the idea relies on what we learned in school that says: assume L is a line in the plane and Q is a point not on the line L. To find the distance between the point Q and L, we draw a perpendicular line, say H, from Q to the line L, then H intersects L in a point A. Hence A is on the line L and it is the nearest point to Q). Now, let $B = \{w_1, ..., w_k\}$ be an orthogonal basis of D. We know that $h = Q - (\Sigma_{j=1}^{k} \frac{<Q,w_j>}{|w_j|^2} w_j)$ is orthogonal (perpendicular) to every w_j . Thus $d = \Sigma_{j=1}^{k} \frac{<Q,w_j>}{|w_j|^2} w_j \in D$ is such point (vector) in D where $|Q-d|$ is minimum. Let $D = span\{w_1 = (1,0,0,0,-4), w_2 = (4,1,1,1,1)\}$. Then D is a subspace of $\mathbb{R}^5$. Given $Q = (2,2,2,2,0)$ is not in D. Find $d \in D$ such that $|Q-d|$ is minimum. Use the normal dot product on $\mathbb{R}^5$. [Hint: to minimize the calculation, w_1, w_2 are already orthogonal.]

Answer:

$$d = \Sigma_{j=1}^{2} \frac{< Q, w_j >}{|w_j|^2} w_j$$

$$= \frac{< Q, w_1 >}{|w_1|^2} w_1 + \frac{< Q, w_2 >}{|w_2|^2} w_2$$

$$= \frac{< (2,2,2,2,0), (1,0,0,0,-4) >}{17} w_1 + \frac{< (2,2,2,2,0), (4,1,1,1,1) >}{20} w_2$$

$$= \frac{2}{17} w_1 + \frac{14}{20} w_2$$

$$= (\frac{248}{85}, \frac{14}{20}, \frac{14}{20}, \frac{14}{20}, \frac{39}{170})$$

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5. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation, $B = \{(5,1),(4,1)\}$ be a basis for $\mathbb{R}^2$, and $C = \{(1,0,1),(0,1,1),(0,-1,0)\}$ is a basis for $\mathbb{R}^3$, Given $T(5,1) = (-1,0,1)$ and $T(4,1) = (0,0,1)$.

(i) Find $[T]_{B,C}$.

Answer: $[T]_{B,C} = K^{-1}M'Q^{-1}$, where

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}, M' = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$$

Therefore, $[T]_{B,C} = \begin{bmatrix} -1 & 4 \\ 1 & -3 \\ 1 & -3 \end{bmatrix}$

(ii) Use (i) and find $T(7,8)$

Answer:

$$[T(7,8)]_{B,C} = \begin{bmatrix} -1 & 4 \\ 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 25 \\ -17 \\ -17 \end{bmatrix}$$

$$\Rightarrow T(7,8) = 25(1,0,1) - 17(0,1,1) - 17(0,-1,0) = (25,0,8)$$

(iii) Find the standard matrix presentation of T

Answer:

$$M_T = M'Q^{-1} = \begin{bmatrix} -1 & 4 \\ 0 & 0 \\ 0 & 4 \end{bmatrix}$$

(iv) Use (iii) and again find $T(7,8)$.

Answer:

$$M_T = M'Q^{-1} = \begin{bmatrix} -1 & 4 \\ 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \\ 8 \end{bmatrix}$$

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MTH 512, Home Work III

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QUESTION 1. Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

(i) Find $A \otimes B$

$$A \otimes B = \begin{pmatrix} 2B & 2B \\ 1B & 2B \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 4 & 2 & 0 & 4 & 2 \\ 0 & 0 & -4 & 0 & 0 & -4 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 2 & 1 & 0 & 4 & 2 \\ 0 & 0 & -2 & 0 & 0 & -4 \end{pmatrix}$$

(ii) Find $|A \otimes B|$

$|A \otimes B| = |A|^m \, |B|^n$, $n = 2$, $m = 3$

$|A| = 4 - 2 = 2$

$|B| = -4$

$|A \otimes B| = (2)^3 (-4)^2 = 128$

(iii) Find $Trace(A \otimes B)$

$Trace(A \otimes B) = Trace(A)\, Trace(B) = (4)(1) = 4$

QUESTION 2. (a) Let $T : V \to V, L : W \to W$ be R-homomorphisms such that α is an eigenvalue of T, and β is an eigenvalue L. Prove that $\alpha\beta$ is an eigenvalue of $T \otimes L$. [hint: note that $T \otimes L(v \otimes w) = T(v) \otimes L(w)$]

$T(v) = \alpha v$ for some $v \in V$.

for some v not = 0_V

$L(w) = \beta w$ for some $w \in W$.

For some w not = 0_W

$(T \otimes L)(v \otimes w) = T(v) \otimes L(w) = \alpha v \otimes \beta w = \alpha\beta (v \otimes w)$

(b) Let $T : R^3 \to R^3, L : R^4 \to R^4$ be R-homomorphisms such that $-3, -3, -7$ are the eigenvalues of T, and $4, 1, 2, 3$ are the eigenvalues of L. If $E_{-3}(T) = span\{(1, 0, 1), (-1, 1, -1)\}$ and $E_4(L) = span\{(1, 0, 1, 0)\}$. In view of (a), find $E_{-12}(T \otimes L)$

I claim if $-3, -3, -6$ are the eigenvalues of T, then I cannot find $E_{-12}(T \otimes L)$ unless more information is provided, why?

{e_1,e_2, e_3} standard basis of R^3, {b_1, b_2, b_3, b_4} standard basis of R^4

$E_{-12}(T \otimes L) = Span\{(1,0,1,0,0,0,0,0,1,0,1,0), (-1,0,-1,0,1,0,1,0,-1,0,-1,0)\}$.

If the eigenvalues are $-3, -3, 6$ then we will have 2 possibilities for $E_{-12}(T \otimes L)$ as $-3 \times 4 = -12$ and $2 \times -6 = -12$.

Use the language of tensor E_{-12} = span{e_1(tensor)b_1 + e_1(tensor)b_3 + e_3(tensor)b_1 + e_3(tensor)b_3,   -e_1(tensor)b_1 - e_1(tensor)b_3 + e_2(tensor)b_1 + e_2(tensor)b_3 - e_3(tensor)b_1 - e_3(tensor)b_3}

(c) Let $T : R^2 \to R^2, L : R^2 \to R^2$ be R-homomorphisms such that $T(a,b) = (2a+4b, -a-2b)$ and $L(a,b) = (-a+b, a-b)$. Find a basis for $Range(T \otimes L)$ and a basis for $Ker(T \otimes L)$. [Hint: let $\{f_1, f_2, f_3, f_4\}$ be the standard basis of R^4, define the colinear $H : R^4 \to R^4$ with the standard matrix presentation $M = [H(f_1), H(f_2), H(f_3), H(f_4)]$, where $H(f_1) = T(e_1) \otimes L(e_1), H(f_2) = T(e_1) \otimes L(e_2), ..., H(f_4) = T(e_2) \otimes L(e_2)$. Now find the range of H and Ker(H). Assume $Range(H) = span\{(3,7,2,1)\}$. Then $Range(T \otimes L) = span\{3e_1 \otimes e_2 + 7e_1 \otimes e_2 + 2e_2 \otimes e_1 + e_2 \otimes e_2\}$].

$T \otimes L : \mathbb{R}^2 \otimes \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \otimes \mathbb{R}^2$

$T \otimes L : (a_1, a_2) \otimes (b_1, b_2) \to T(a_1, a_2) \otimes L(b_1, b_2)$

H is the co-linear $H : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$T(e_1) = T(1,0) = (2, -1)$, $L(e_1) = L(1,0) = (-1, 1)$

$T(e_2) = T(0,1) = (4, -2)$ $L(e_2) = L(0,1) = (1, -1)$

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$H(f_1) = T(e_1) \otimes L(e_1) = (2,-1) \otimes (-1,1) = (-2, 2, 1, -1)$

$H(f_2) = T(e_1) \otimes L(e_2) = (2,-1) \otimes (1,-1) = (2, -2, -1, 1)$

$H(f_3) = T(e_2) \otimes L(e_1) = (4,-2) \otimes (-1,1) = (-4, 4, 2, -2)$

$H(f_4) = T(e_2) \otimes L(e_2) = (4,-2) \otimes (1,-1) = (4, -4, -2, 2)$

$$M_H = \begin{pmatrix} -2 & 2 & -4 & 4 \\ 2 & -2 & 4 & -4 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$Range(H) = Span\{(1, -1, 2, -2)\}$

$Range(T \otimes L) = span\{e_1 \otimes e_1 + -e_1 \otimes e_2 + 2e_2 \otimes e_1 + -2e_2 \otimes e_2\}$

To find the $Ker(T \otimes L)$ we solve $\left(\begin{array}{cccc|c} -2 & 2 & -4 & 4 & 0 \\ 2 & -2 & 4 & -4 & 0 \\ 1 & -1 & 2 & -2 & 0 \\ -1 & 1 & -2 & 2 & 0 \end{array}\right)$

$Ker(H) = Span\{(1,1,0,0), (-2,0,1,0), (2,0,0,1)\}$

$Ker(T \otimes L) = Span\{e_1 \otimes e_1 + e_1 \otimes e_2, -2e_1 \otimes e_1 + e_2 \otimes e_1, 2e_1 \otimes e_1 + e_2 \otimes e_2\}$

QUESTION 3. Let D be a subspace of a real inner product space V and assume $2 \leq dim(V) = n$. Then the orthogonal space of D is denoted by $D^{\perp}$ and $D^{\perp} = \{w \in V \mid < w, d >= 0 \text{ for every } d \in D\}$. It is clear that if D has a basis $B = \{v_1, ..., v_k\}$, then $f \in D^{\perp}$ if and only if $< f, v_i >= 0$ for every $1 \leq i \leq k$.

(i) Prove that $D \cap D^{\perp} = \{O_V\}$[hint: Trivial, it should not exceed 2 lines]

assume $\exists\, v \neq O_v \in D \cap D^{\perp} \Rightarrow v \in D,\ v \in D^{\perp}$

By definition of $D^{\perp}$ $\langle v, v \rangle = 0 \iff v = O_v$!! Contradiction.

(ii) Prove that $D + D^{\perp} = V$ and $dim(D + D^{\perp}) = dim(D) + dim(D^{\perp})$ (recall that $H + F = \{h + f \mid h \in H$ and $f \in F\}$)[hint: One way, assume $B = \{v_1, ..., v_k\}$ is an ORTHOGONAL basis for D. Then B can be extended to a basis for V, how? choose v_{k+1} not in D, then $v_{k+1}, v_1, v_2, ..., v_k$ are independent, continue this process, choose v_{k+2} not in $span\{v_1, ..., v_k, v_{k+1}\}$, then $v_{k+2}, v_{k+1}, v_1, ..., v_k$ are independent. By continuing this process, we get a basis $K = \{v_1, ..., v_k, v_{k+1}, ..., v_n\}$ for V. Now for each $k + 1 \leq i \leq n$, use the (modified) Gram-Schmidt process(i.e., just use the orthogonal basis of D, see(b) in Question 4) and form the elements $w_{k+1}, ..., w_n$, where $w_i = v_i - (\sum_{j=1}^{k} \frac{<v_i,v_j>}{|v_j|^2} v_j)$. Note that each w_i is orthogonal to v_j for every $1 \leq j \leq k$. Then $W = \{v_1, .., v_k, w_{k+1}, ..., w_n\}$ form a basis for V (note that since K is a basis for V, by construction, W is a basis for V!!). By construction, $D^{\perp} = span\{w_{k+1}, ..., w_n\}$. Since $D \cap D^{\perp} = \{o_v\}$, it is clear $dim(D + D^{\perp}) = dim(D) + dim(D^{\perp}) = n$. Thus $V = D + D^{\perp}$]

To prove $D + D^{\perp} = V$ we prove that $\forall\, v \in V$, $v = w + d$ s.t. $d \in D$ and $w \in D^{\perp}$.

$B = \{v_1, ..., v_k\}$ is orthogonal basis for D.

Take any $v \in V$. Then $v_0 = v - \frac{\langle v, v_1 \rangle}{|v_1|^2} v_1 - \frac{\langle v, v_2 \rangle}{|v_2|^2} v_2 - \dots - \frac{\langle v, v_k \rangle}{|v_k|^2} v_k$

then v_0 is orthogonal to v_i for $1 \leq i \leq k$. $\Rightarrow v_0 \in D^{\perp}$

$$v = v_0 + \underbrace{\frac{\langle v, v_1 \rangle}{|v_1|^2} v_1 + \frac{\langle v, v_2 \rangle}{|v_2|^2} v_2 + \dots + \frac{\langle v, v_k \rangle}{|v_k|^2} v_k}_{\in D}$$

(this is a linear combination of the basis)

$\Rightarrow V = D + D^{\perp}$

$$dim(D + D^{\perp}) = dim(D) + dim(D^{\perp}) - \underbrace{dim(D \cap D^{\perp})}_{D \cap D^{\perp} = \{O_v\}}$$

$dim(\{O_v\}) = 0$.

$\Rightarrow dim(D + D^{\perp}) = dim(D) + dim(D^{\perp})$

(iii) We know that $\int_0^1 f(x)\,dx$ is an inner product on P_4. Let $D = span\{x^2, 1\}$. use the hint in (i) and find $D^{\perp}$.

Let $v_1 = 1$. Find the orthogonal projection of x^2 over 1. Translate to $\mathbb{R}^4$.

$|v_1|^2 = \langle 1, 1\rangle = \int_0^1 1\,dx = 1$, $\langle x^2, 1\rangle = \int_0^1 x^2 \cdot 1\,dx = 1/3$

$$v_2 = x^2 - \frac{\langle x^2, 1\rangle}{1} \cdot 1 = x^2 - 1/3$$

$$\langle 1, x^2 - 1/3\rangle = \int_0^1 x^2 - 1/3\,dx = \frac{x^3}{3} - 1/3\,x \Big|_0^1 = 0$$

$\Rightarrow \{1, x^2 - 1/3\}$ is orthogonal basis of D.

Take $x^3 \notin span\{1, x^2 - 1/3\}$

$\langle x^3, 1\rangle = 1/4$, $\langle x^3, x^2 - 1/3\rangle = 1/12$, $\langle x^2 - 1/3, x^2 - 1/3\rangle = 4/45$

$$v_3 = x^3 - \frac{1/4}{1} \cdot 1 - \frac{1/12}{4/45}(x^2 - 1/3)$$

$$= x^3 - 1/4 - \frac{15}{16}x^2 + \frac{5}{16} = x^3 - \frac{15}{16}x^2 + \frac{1}{16}$$

we can check $\langle v_1, v_3\rangle = 0$ and $\langle v_2, v_3\rangle = 0$

Take $x \notin span\{v_1, v_2, v_3\}$

$\langle x, 1\rangle = 1/2$, $\langle x, v_2\rangle = 1/12$

$$v_4 = x - \frac{1/2}{1} \cdot 1 - \frac{1/12}{4/45}(x^2 - 1/3) = x - \frac{15}{16}x^2 - \frac{3}{16}$$

we can check $\langle v_1, v_4\rangle = 0$ and $\langle v_2, v_4\rangle = 0$

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$$D^{\perp} = Span\{x^3 - \frac{15}{16}x^2 + \frac{1}{16}, x - \frac{15}{16}x^2 - \frac{3}{16}\}$$

you used the proof to do the calculation, and that is fine. See the solution by Jamila, simpler Calculations

(iv) (nice, no need for Gram schmidt Thm) Let $D = span\{Q_1 = (1,0,0,1,0), Q_2 = (0,1,1,1,0)\}$. It is clear dim(D) = 2. Use the normal dot product on R^5 and find $D^{\perp}$.[Hint: Matrix multiplication is a dot product of 2 vectors!. So construct the matrix M, 2×5, i.e., Q_1 = first row, Q_2 = second row. Then find Null(M), i.e., the solution to the homogeneous system $MX = 0$, then $D^{\perp} = span\{basis\ of\ Null(M)\}$]

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

$x_2 = -x_3 - x_4$

$x_1 = -x_4$

x_3, x_4, x_5 free variables

$Null(M) = span\{(0,-1,1,0,0), (-1,-1,0,1,0), (0,0,0,0,1)\} = D^{\perp}$

(v) Use the "mimic dot product" on P_5. Let $D = span\{x^4 + x, x^3 + x^2 + x\}$. Find $D^{\perp}$. [hint: Translate to R^5, use the normal dot product on R^5. Stare at (iv), translate the answer of (iv) to P_5. Done]

Translate D to $\mathbb{R}^5$. $\tilde{D} = span\{(1,0,0,1,0), (0,1,1,1,0)\}$

$\tilde{D}^{\perp} = Null(M)$ (M from the previous part)

$\tilde{D}^{\perp} = span\{(0,-1,1,0,0), (-1,-1,0,1,0), (0,0,0,0,1)\}$

$D^{\perp} = span\{-x^3 + x^2, -x^4 - x^3 + x, 1\}$

QUESTION 4. (a) Let V be a normed vector space over R. Prove that $\frac{||x+y||+||x-y||}{2} \le ||x|| + ||y||$.

$$\frac{\|x+y\| + \|x-y\|}{2} \le \frac{\|x\|}{2} + \frac{\|y\|}{2} + \frac{\|x\|}{2} + \frac{\|-y\|}{2} = \|x\| + \|y\|$$

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(b) Let V be a real inner product vector space and $v_1, ..., v_k$ be nonzero pairwise othogonal vectors (points) in V. Choose $Q \notin span\{v_1, ..., v_k\}$ and let $h = Q - (\sum_{j=1}^{k} \frac{<Q,v_j>}{|v_j|^2} v_j)$. Prove that h is orthogonal to every v_i, $1 \le i \le k$. [Hint: Routine calculations using the definition of inner product]

Choose any i. Show $\langle h, v_i \rangle = 0$

$$\langle h, v_i \rangle = \langle Q - \sum_{j=1}^{k} \frac{\langle Q, v_j \rangle}{|v_j|^2} v_j \,, v_i \rangle$$

$$= \langle Q, v_i \rangle - \underbrace{\sum_{j=1}^{k} \frac{\langle Q, v_j \rangle}{|v_j|^2} \langle v_j, v_i \rangle}_{0 \text{ for } j \ne i}$$

$$= \langle Q, v_i \rangle - \frac{\langle Q, v_i \rangle}{\cancel{|v_i|^2}} \cancel{\langle v_i, v_i \rangle} = 0$$

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(c) V be a real inner product vector space, and $D = span\{v_1, ..., v_k\}$, such that $dim(D) = k < dim(V)$. Choose a point Q in $V - D$. Then the distance between Q and D is denoted by $d(Q, D) = min\{|Q - d| \mid d \in D\}$. Question? how do we find $d \in D$ such that $|Q - d|$ is minimum. Answer: (the idea relies on what we learned in school that says: assume L is a line in the plane and Q is a point not on the line L. To find the distance between the point Q and L, we draw a perpendicular line, say H, from Q to the line L, then H intersects L in a point A. Hence A is on the line L and it is the nearest point to Q). Now, let $B = \{w_1, ..., w_k\}$ be an orthogonal basis of D. We know that $h = Q - (\sum_{j=1}^{k} \frac{<Q,w_j>}{|w_j|^2} w_j)$ is orthogonal (perpendicular) to every w_j. Thus $d = \sum_{j=1}^{k} \frac{<Q,w_j>}{|w_j|^2} w_j \in D$ is such point (vector) in D where $|Q - d|$ is minimum.

Let $D = span\{w_1 = (1, 0, 0, 0, -4), w_2 = (4, 1, 1, 1, 1)\}$. Then D is a subspace of R^5. Given $Q = (2, 2, 2, 2, 0)$ is not in D. Find $d \in D$ such that $|Q - d|$ is minimum. Use the normal dot product on R^5. [Hint: to minimize the calculation, w_1, w_2 are already orthogonal.]

$$d = \frac{\langle Q, w_1 \rangle}{|w_1|^2} w_1 + \frac{\langle Q, w_2 \rangle}{|w_2|^2} w_2$$

$$\langle Q, w_1 \rangle = (2,2,2,2,0) \cdot (1,0,0,0,-4) = 2$$

$$\langle Q, w_2 \rangle = (2,2,2,2,0) \cdot (4,1,1,1,1) = 8 + 2 + 2 + 2 = 14$$

$|w_1|^2 = (1)^2 + (-4)^2 = 17$

$|w_2|^2 = (4)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2 = 20$

$$d = \frac{2}{17}(1,0,0,0,-4) + \frac{14}{20}(4,1,1,1,1)$$

$$= \left(\frac{248}{85}, \frac{7}{10}, \frac{7}{10}, \frac{7}{10}, \frac{39}{170}\right)$$

QUESTION 5. Let $T : R^2 \to R^3$ be a linear transformation, $B = \{(5,1),(4,1)\}$ be a basis for R^2, and $C = \{(1,0,1),(0,1,1),(0,-1,0)\}$ is a basis for R^3, Given $T(5,1) = (-1,0,1)$ and $T(4,1) = (0,0,1)$.

(i) Find $[T]_{B,C}$.

$[T]_{B,C} = C^{-1} M Q^{-1}$

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$[T]_{B,C} = \underbrace{\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}}_{3\times3} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}}_{3\times2}}_{\text{call it } L} \underbrace{\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}}_{2\times2} = \begin{pmatrix} -1 & 4 \\ 1 & -3 \\ 1 & -3 \end{pmatrix}$$

call it L

note M_T = LQ^{-1}

(ii) Use (i) and find $T(7,8)$

$$[T(7,8)]_{B,C} = \begin{pmatrix} -1 & 4 \\ 1 & -3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 25 \\ -17 \\ -17 \end{pmatrix}$$

$$T(7,8) = 25\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 17\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 17\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ 0 \\ 8 \end{pmatrix}$$

(iii) Find the standard matrix presentation of T

$$[T(1,0)]_{B,C} = \begin{pmatrix} -1 & 4 \\ 1 & -3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$T(1,0) = -1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(0,1)]_{B,C} = \begin{pmatrix} -1 & 4 \\ 1 & -3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -3 \end{pmatrix}$$

$$T(0,1) = 4\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow M_T = \begin{pmatrix} -1 & 4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(iv) Use (iii) and again find $T(7,8)$.

$$T(7,8) = M_T \cancel{Q^{-1}}\begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\cancel{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 25 \\ 0 \\ 8 \end{pmatrix}$$

Since M_T is the standard matrix, we have T(7, 8) = M_T(7,8)^T = (25, 0, 8). NO NEED for Q^{-1} !!

note that M_T = LQ^{-1}, see above