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MTH 512, Home Work I, Warm Up

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QUESTION 1. Let V be a vector over F, we know that if $0 \in F$ and $v \in V$, then $0.v = O_v \in V$. Let $v \in V$. Prove that $a.O_v = O_v$ for every $a \in F$.

we know $v + -v = O_v$ where $-v$ is the additive inverse of v.

$a \cdot O_v = a(v + -v) = a \cdot v + a \cdot (-v) = a \cdot v + -a \cdot v$

$= (a + -a) \cdot v = 0 \cdot v = O_v$

$\therefore \; a \cdot O_v = O_v$

QUESTION 2. Let $W = \{f(x) \in P_5 \mid f(-1) = 0\}$. Prove that W is a subspace of of P_5. Find a basis for W.

$f(x) = a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5$

$f(-1) = a_1(-1)^4 + a_2(-1)^3 + a_3(-1)^2 + a_4(-1) + a_5 = 0$

$a_1 - a_2 + a_3 - a_4 + a_5 = 0 \Rightarrow a_1 = a_2 - a_3 + a_4 - a_5$

$W = \{(a_2 - a_3 + a_4 - a_5)x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 \mid a_2, a_3, a_4, a_5 \in \mathbb{R}\}$

$= \{a_2(x^4 + x^3) + a_3(x^2 - x^4) + a_4(x^4 + x) + a_5(1 - x^4) \mid a_2, a_3, a_4, a_5 \in \mathbb{R}\}$

$= \text{span}\{x^4 + x^3, \; x^2 - x^4, \; x^4 + x, \; 1 - x^4\}$

W is the span of finite number of vectors, so W is a subspace.

$\dim(W) = 4.$

Check independent?

QUESTION 3. (a) Convince me that $W = \{abx^2 + bx + a \mid a, b \in \mathbb{R}\}$ is not a subspace of P_3.[hint: show that one of the subspace axioms fails]

$W = \{abx^2 + bx + a \mid a, b \in \mathbb{R}\}$

let $\alpha \in \mathbb{R}$

$\alpha(abx^2 + bx + a) = \alpha abx^2 + \alpha bx + \alpha a \notin W$

why? it doesn't have the form $abx^2 + bx + a$

W is not closed under scalar multiplication.

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Give example

$\alpha = 1, \alpha = 2$

$\in W$

$\notin$ if $\alpha \neq \alpha$

(b) Convince me that $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(a,b,c) = (ab, c+a)$ is not a linear transformation, i.e. T is not an $\mathbb{R}$-homomorphism.

Let $v_1 = (a_1, b_1, c_1)$, $v_2 = (a_2, b_2, c_2) \in \mathbb{R}^3$

$T(v_1 + v_2) = T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$

$= ((a_1+a_2)(b_1+b_2), (c_1+c_2)(a_1+a_2))$

$= (a_1 b_1 + a_1 b_2 + a_2 b_2 + a_2 b_1, (c_1+c_2)(a_1+a_2))$

$\neq \underbrace{(a_1 b_1, c_1 + a_1)}_{T(v_1)} + \underbrace{(a_2 b_2, c_2 + a_2)}_{T(v_2)}$

$\therefore T(v_1 + v_2) \neq T(v_1) + T(v_2)$

for some a1, b1, c1, a2, b2, c2 this might be true. Again disprove by example/ take particular values for a1, b1,... c2

c) Convince me that $W = \{ax^3 + (a+b+1)x + b \mid a, b \in R\}$ is not a subspace pf P_4.

Let $a=0$ and $b=0$.

$0 \cdot x^3 + (0+0+1)x + 0 = x$

$\therefore 0 \notin W$, $\nexists\, a, b \in \mathbb{R}$ s.t. $ax^3 + (a+b+1)x + b = 0$.

$\therefore W$ is not a subspace of P_4.

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QUESTION 4. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -2 & -4 \\ 3 & 3 & 6 \end{bmatrix}$, and $W = \left\{ B \in \mathbb{R}^{3\times 3} \mid AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$. Prove that W is a subspace of $\mathbb{R}^{3\times 3}$. Find a basis for W. [hint: using the axioms of subspace might be easier than the span-method (so it is your choice). To find a basis, you need to scratch your head a little. For calculations, you are allowed to use the tools that I will add on I-learn.]

Let $B_1, B_2 \in W$. Then $AB_1 = O_W$ and $AB_2 = O_W$.

$A(B_1 + B_2) = AB_1 + AB_2 = O_W + O_W = O_W$.

$\therefore W$ is closed under addition.

Let $\alpha \in \mathbb{R}$, $B \in W$. Then $AB = O_W$.

$A(\alpha B) = \alpha \cdot AB = \alpha \cdot (AB) = \alpha \cdot O_W = O_W$

$\therefore W$ is closed und scalar multiplication.

$\therefore W$ is a subspace of $\mathbb{R}^{3\times 3}$.

$$AB = \begin{pmatrix} 1 & 1 & 2 \\ -2 & -2 & -4 \\ 3 & 3 & 6 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}$$

$$= \begin{pmatrix} b_1 + b_4 + 2b_7 & b_2 + b_5 + 2b_8 & b_3 + b_6 + 2b_9 \\ -2b_1 - 2b_4 - 4b_7 & -2b_2 - 2b_5 - 4b_8 & -2b_3 - 2b_6 - 4b_9 \\ 3b_1 + 3b_4 + 6b_7 & 3b_2 + 3b_5 + 6b_8 & 3b_3 + 3b_6 + 6b_8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$b_1 + b_4 + 2b_7 = 0$, $\quad b_2 + b_5 + 2b_8 = 0$, $\quad b_3 + b_6 + 2b_9 = 0$

$b_1 = -b_4 - 2b_7 \quad b_2 = -b_5 - 2b_8 \quad b_3 = -b_6 - 2b_9$

$$W = \left\{ \begin{pmatrix} -b_4 - 2b_7 & -b_5 - 2b_8 & -b_6 - 2b_9 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix} \middle| \; b_4, b_5, b_6, b_7, b_8, b_9 \in \mathbb{R} \right\}$$

$$W = \left\{ b_4 \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b_5 \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b_6 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right.$$

$$\left. + b_7 \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + b_8 \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + b_9 \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; \begin{matrix} b_4, b_5, b_6 \\ b_7, b_8, b_9 \end{matrix} \in \mathbb{R} \right\}$$

$$W = \operatorname{Span} \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

This set spans the space W and it is independent so it forms a basis of W.

QUESTION 5. Construct a basis, say B, for P_4 such that $deg(f) = 3$ for every $f \in B$.

Let $B = \{x^3, x^3+x^2, x^3+x^2+x, x^3+x^2+x+1\}$

Translate this into $\mathbb{R}^4$

$\tilde{B} = \{(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)\}$.

By the tools on i learn the vector in $\tilde{B}$ are linearly independent so

$\mathrm{span}(\tilde{B}) = \mathbb{R}^4$

$\Rightarrow$ B spans P_4

QUESTION 6. (very nice result) Let $T : V \to W$ be an $\mathbb{F}$-homomorphism. Given $T(v_0) = w_0$ for some $v_0 \in V$ and $w_0 \in W$. Let D be the set of all elements, say v, in the domain V such that $T(v) = w_0$. Prove that $D = \{a + v_0 \mid a \in Ker(T)\}$.

Show $D = \{a + v_0 \mid a \in Ker(T)\}$

- Show $D \subset \{a + v_0 \mid a \in Ker(T)\}$

Take $v \in D$. $v = (v + -v_0) + v_0$

$T(v + -v_0) = T(v) - T(v_0) = w_0 - w_0 = 0$

Then $(v + -v_0) \in Ker(T)$

Let $a = (v + -v_0)$

$\Rightarrow$ $v \in \{a + v_0 \mid a \in Ker(T)\}$.

- show $D \supset \{a + v_0 \mid a \in Ker(T)\}$.

Take $a + v_0$ s.t. $a \in Ker(T)$.

$T(a + v_0) = T(a) + T(v_0) = 0 + w_0$

$\Rightarrow$ $a + v_0 \in D$.