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MTH 221, Final Exam

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Score = $\frac{52}{52}$ Excellent

QUESTION 1. (Each = 1.5 points, Total = 15 points)

(i) Let A be a 3×3 matrix, where $A = \begin{bmatrix} -1 & 0 & 0 \\ -9 & 6 & b \\ 5 & b & 3 \end{bmatrix}$. Given $\alpha = 2$ is an eigenvalue of A. Then $b =$

(a) 4 or -4 (b) 2 or -2 (c) 3 or -3 (d) 3 only

(ii) Given $D = span\{(1,0,0),(1,1,-1)\}$. Use the normal dot product on D. Then $D^{\perp}$ (i.e., the orthogonal space of D) =

(a)$\{(0,1,-1)\}$ (b) span$\{(0,1,1)\}$ (c) $\{(0,1,1)\}$ (d) span$\{(1,0,0),(0,1,-1)\}$

(iii) consider the "mimic dot product" on $R^{2\times 2}$, i.e., $< A, B >= Trace(B^T A)$. Given $A = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}$. Then the distance between A and B is

(a)$\sqrt{7}$ (b) 5 (c) 0 (d) 3

(iv) Let $A = \begin{bmatrix} 1 & b & -6 \\ -1 & -2 & 12 \\ -1 & -b & c \end{bmatrix}$. Consider the system of linear equations

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1.987 \\ 3.7 \\ 2.23 \end{bmatrix}$$

. Then the system will have unique solution in one of the cases:

(a) $b = 77.98, c = 6$ (b) $b = 2.65,\ c = -3.23$ (c) $b = 2, c = 7$ (d) $b = 0.5,\ c = 6$

(v) Let $T : P_3 \to R^{2\times 2}$ be a linear transformation such that $T(ax^2 + bx + c) = \begin{bmatrix} a - 2b + c & -a + 2b + c \\ 2a - 4b + 2c & 0 \end{bmatrix}$.
Then a basis for $Ker(T) = Z(T) = Nul(T)$ is

(a) $\{2x^2 + x\}$ (b) $\{2x + 1\}$ (c) $\{2x^2 + 1, x^2 + 2x\}$ (d) $\{x^2, 2x\}$

(vi) Let A be a 4×4 DIAGNOLIZABLE matrix such that $2, 5$ are the eigenvalues of A. Given $E_2 = span\{(3,3,0,-1)\}$. Then $C_\alpha(A) =$

(a) $(\alpha-2)(\alpha-5)^3$ (b) $(\alpha-2)^2(\alpha-5)^2$
(c) $(\alpha-2)^3(\alpha-5)$ (d) more information is needed

(vii) Let A, B be 2×2 matrices, such that $1, 2$ are the eigenvalues of A and $-2, -1$ are the eigenvalues of B. We know that $A \otimes B$ is a 4×4 matrix. Then $| A \otimes B | =$

(a) 16 (b) 0 (c) 4 (d) -4

(viii) Given $1, -1$ are the eigenvalues of a 2×2 matrix, A. Then $|A^3 + A^{-1} + 4I_2| =$

(a) 2 (b) -2 (c) 14 (d) 12

(ix) Assume that the normal dot product is defined on R^4. Given $\{Q,\ F,\ (1,0,0,3)\}$ is an orthogonal basis for a subspace W of R^4, for some points Q, F in R^4. Given $(11,\ 23,\ 51,\ 13) \in W$. Then $(11,23,51,13) = c_1Q + c_2F + c_3(1,0,0,3)$. Then $c_3 =$

(a)50 (b) 10 (c) (d) 5

(x) Let A be a 2×2 matrix with eigenvalues $2, 3$. Given $E_2 = span\{(2,4)\}$ and $E_3 = span\{(-2,-3)\}$. Let D be the solution set to the system $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $D =$

(a) $\{(-2,-3),(2,4)\}$ (b) span$\{(-2,-3),(2,4)\}$ (c) $\{(0,1)\}$ (d) $\{(0,0)\}$

$\alpha = 0, 1, 2$

all repeated once.

QUESTION 2. (10 points) Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$.

(i) **(3 points)** Find all eigenvalues of A.

$$C_A(\alpha) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ -1 & \alpha & 2 \\ 0 & -1 & \alpha-3 \end{bmatrix}$$

$$(-1)^{1+1}(\alpha) \begin{vmatrix} \alpha & 2 \\ -1 & \alpha-3 \end{vmatrix} = \alpha(\alpha(\alpha-3)+2) = 0$$

$$\alpha[\alpha^2 - 3\alpha + 2] = 0$$

$$\alpha(\alpha-2)(\alpha-1) = 0$$

(ii) **(4 points)** For each eigenvalue a of A, find $E_a(A)$.

$$E_0 = \begin{array}{c} \begin{matrix} a & b & c & \end{matrix} \\ \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right] \end{array}$$

$c \in \mathbb{R}$

$-a + 2c = 0 \Rightarrow a = 2c$

$-b - 3c = 0 \Rightarrow b = -3c$

$= \{2c, -3c, c \mid c \in \mathbb{R}\}$

$= \text{span}\{(2, -3, 1)\}$

$$E_1 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_2 + R_3 \to R_3}$$

$$\begin{array}{c} \begin{matrix} a & b & c & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad c \in \mathbb{R}$$

$a = 0$

$b + 2c = 0$

$b = -2c$

$= \{0, -2c, c \mid c \in \mathbb{R}\}$

$= \text{span}\{(0, -2, 1\}$

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(iii) **(3 points)** If A is diagnolizable, find an invertible matrix Q and a diagonal matrix D such that $QDQ^{-1} = A$

A is diagnolizable since: $\dim(E_0) = 1$

$\dim(E_1) = 1$

$\dim(E_2) = 1$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad Q = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

QUESTION 3. Let $T : P_4 \to P_3$ such that $T(ax^3+bx^2+cx+d) = (a+b+c+d)x^2+(-a-b-c)x+-a-b-c-d$

(i) **(2 points)** Convince me that T is a linear transformation.

Each coordinate in the co-domain (T) is a linear combination of a, b, c, d

(ii) **(4 points)** Find all points in the domain (i.e., in P_4) such that $T(ax^3+bx^2+cx+d) = 2x^2+3x-2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ -1 & -1 & -1 & 0 & 3 \\ -1 & -1 & -1 & -1 & -2 \end{array}\right] \xrightarrow[R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$-R_2+R_1 \to R_1$

$b, c \in R$ $$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$a+b+c=-3$

$a = -3-b-c$

$d = 5$

$= \{-3-b-c, b, c, 5 \mid b, c \in R\}$

$= \{(-3-b-c)x^3 + bx^2 + cx + 5 \mid b, c \in R\}$

(iii) **(3 points)** Find a basis for the Range(T).

$$R_1+R_2 \to R_2 \qquad R_1+R_3 \to R_3 \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis (Range (L)) $= \{(1,-1,-1), (1,0,-1)\}$

Basis (range (T)) $= \{(x^2-x-1), (x^2-1)\}$

(iv) **(2 points)** Find $Z(T) = Ker(T) = Null(T)$ and write it as span of independent points.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$b, c \in R$

$a+b+c=0$

$d=0$

$a=-b-c$

$= \{(-b-c, b, c, 0) \mid b, c \in R\}$

$= \{b(-1,1,0,0), c(-1,0,1,0)\}$

$= \text{span}\{-x^3+x^2, -x^3+x\}$

$B = 2 \times 3$ $\qquad$ $4x + 1\alpha$

QUESTION 4. (4 points) Let B be a basis for R^3 and $C = \{(2,2),\ (1,2)\}$ is a basis for R^2. Let $T : R^3 \to R^2$ be a linear transformation such that the coordinate matrix presentation of T with respect to B and C is $[T]_{B,C} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix}$. Find the standard matrix presentation of T.

$T(1,0,0) = 1(2,2) + -1(1,2) = (2,2) + (-1,-2)$

$= (1,0)$

$T(0,1,0) = 1(2,2) + -1(1,2) = (2,2) + (-1,-2)$

$= (1,0)$

$T(0,0,1) = 2(2,2) + 1(1,2) = (4,4) + (1,2)$

$= (5,6)$

$$M = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

QUESTION 5. (3 points) Let $T : P_2 \to R$ such that $T(x+3) = 10$ and $T(6) = 12$. Find $T(4x)$.

$4T(x+3) = 40$

$2T(6) = 24$

$T(4x) = 4T(x+3) - 2T(6)$

$= 40 - 24 = 16$

QUESTION 6. (3 points) Use the "integral inner product" on P_3. Given $D = span\{x+1, x\}$ is a subspace of P_3. Use Gram-Schmidt algorithm and find an orthogonal basis for D.

$W_1 = x + 1$

$$W_2 = x - \frac{\int_0^1 x(x+1)\,dx}{\left[\sqrt{\int_0^1 (x+1)^2}\right]^2}(x+1)$$

$$\int_0^1 x^2 + x\,dx = \left[\frac{x^3}{3} + \frac{1}{2}x^2\right]_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

$(x+1)^2 = (x+1)(x+1) = x^2 + 2x + 1$

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QUESTION 7. (6 points) Given M, N are subspaces of R^4 such that

$$M = span\{(1,1,1,1), (-1,0,-1,-1), (-1,-1,-1,0)\}$$

and $N = span\{(0,1,0,0), (0,0,1,0)\}$, where $dim(M) = 3$ and $dim(N) = 2$.

a) Find a basis for $M + N$.

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} -R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ -R_1 + R_4 \to R_4 \end{matrix} \quad \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Basis $(M + N) = \{(1,1,1,1), (-1,0,-1,-1), (-1,-1,-1,0), (0,0,1,0)$

b) Find a basis for $M \cap N$.

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_4 + R_1 \to R_1} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$R_2 + R_1 \to R_1$

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$d \in R$

$a + d = 0$

$b + d = 0$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

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