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Exam 2

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Score = $\frac{45}{45}$ Excellent

QUESTION 1. (15 points, each = 1.5 points)

(i) Let $T : P_3 \to P_3$ be a linear transformation such that $T(ax^2 + bx + c) = (a - 3b)x^2 + cx + 2c$. Then a basis for $Ker(T) = Z(T) = Nul(T)$ is

(a) $\{3x + 1\}$ (b) $\{3x^2 + x\}$ (c) $\{x^2 - 3\}$ (d) $\{-3x + 1\}$

(ii) Let T as above. Then a basis for the Range(T) is

(a) $\{x^2 - 3x,\ 1\}$ (b) $\{x^2,\ x + 2\}$ (c) $\{x^2,\ x\}$ (d) $\{x^2,\ 1\}$

(iii) Consider the integral inner product on P_3, where $a = 0$ and $b = 1$. The distance between $f_1(x) = x^2 + 3x + 1$ and $f_2(x) = x^2 + 1$

(a) 9 (b) $\sqrt{3}$ (c) 3 (d) 4.5

(iv) Let $T : P_3 \to P_3$ such that $T(ax^2 + bx + c) = (4a + b + c)x^2 + 5bx + 5c$. Then 5 is an eigenvalue of T. Hence E_5 =

(a) $span\{x^2 + x,\ x^2 + 1\}$ (b) $span\{x^2 - x - 1\}$ (c) $span\{x^2 + x + 1\}$ (d) $span\{1, x\}$

(v) Let A be a 3×3 matrix such that $C_A(\alpha) = (\alpha - 2)^2(\alpha - 3)$. Given $E_2 = span\{(1,0,0)\}$ and $E_3 = span\{(0,2,2)\}$. One of the following statements is true

(a) A is not diagnolizable (b) It is possible that A^{-1} does not exist (c) $Trace(A) = 5$ (d) A^{-1} is diagnolizable

(vi) Given $D = \{\begin{bmatrix} a & -a \\ b & b \end{bmatrix} \mid a, b \in R\}$ is a subspace of $R^{2\times 2}$. Then a basis for D is

(a) $\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\}$ (b) $\{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\}$

(c) $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ (d) $\{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\}$

(vii) Let A be a 2×2 matrix such that $C_A(\alpha) = (\alpha - 2)(\alpha - b)$. Given $|2A - 3I_2| = 5$. Then $|A|$ =

(a) 8 (b) 4 (c) 2 (d) 16

(viii) Consider the normal dot product on R^3. One of the following is an orthogonal basis for R^3

(a) $\{(4,0,0), (0,4,-2), (0,1,2)\}$ (b) $\{(1,0,1), (-1,1,1), (-2,0,2)\}$ (c) $\{(1,0,1), (-1,1,1)\}$ (d) $\{(1,0,-1), (0,1,2), (0,0,1)\}$

(ix) Let A, B be 2×2 matrices such that $1, -1$ are the eigenvalues of A and $2, -2$ are the eigenvalues of B. Then $|A \otimes B|$ = (i.e., find the determinant of the tensor product $A \otimes B$)

(a)4 (b) -4 ✓(c) 16 (d) 9

(x) Consider the integral inner product on P_3, where $a = 0$ and $b = 1$. Given $\{w_1,\ w_2,\ x\}$ is an orthogonal basis for P_3. Hence, we know that $10 = c_1w_1 + c_2w_2 + c_3x$, for some real numbers c_1, c_2, c_3. Then c_3 =

(a) 10 (b) 5 (c) 30 ✓(d) 15

QUESTION 2. (12 points) Given $T : P_4 \to R^{2\times 2}$ such that $T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a-2b & c-d \\ -a+2b & -c+d \end{bmatrix}$ is a linear transformation.

(i) **(3 points)** Find a basis for the range of T

$$T(ax^3+bx^2+cx+d) = \begin{bmatrix} a-2b & c-d \\ -a+2b & -c+d \end{bmatrix} \approx T(a,b,c,d) = (a-2b, c-d, -a+2b, -c+d)$$

$$T(a,b,c,d) = a(1,0,-1,0) + b(-2,0,2,0) + c(0,1,0,-1) + d(0,-1,0,1)$$

$$M_T = \begin{array}{cccc} a & b & c & d \end{array}$$
$$M_T = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad R_1 + R_3 \to R_3 \quad \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$R_2 + R_4 \to R_4 \quad \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Range}(T) = \text{span}\{(1,0,-1,0),\ (0,1,0,-1)\}$$

$$\text{Basis for Range}(T) = \{(1,0,-1,0),\ (0,1,0,-1)\}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$$

✓

(ii) **(4 points)** Find all polynomials in P_4, such that $T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix}$

≈ $T(a,b,c,d) = (2,4,-2,-4)$

~~x(1,0,-1,0) + y(0,1,0,-1~~

$$\begin{array}{cccc|c} a & b & c & d & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & 4 \\ -1 & 2 & 0 & 0 & -2 \\ 0 & 0 & -1 & 1 & -4 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(same operations as (i))

$c - d = 4 \Rightarrow c = 4 + d$ $\quad a - 2b = 2 \Rightarrow a = 2 + 2b$

Soln. set = $\{(2+2b, b, 4+d, d) \mid b, d \in \mathbb{R}\}$

$= \{(2+2b)x^3 + bx^2 + (4+d)x + d \mid b, d \in \mathbb{R}\}$

✓

(iii) **(3 points)** Find a basis for $Z(T) = Ker(T) = Nul(T)$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(same operations as (i))

$c = d$, $a = 2b$

Soln. set = $\{(2b, b, d, d) \mid b, d \in \mathbb{R}\} = \text{span}\{(2,1,0,0), (0,0,1,1)\}$

Basis for $Z(T) = \{(2,1,0,0), (0,0,1,1)\} = \{2x^3 + x^2, x + 1\}$

✓ Excellent

(iv) **(2 points)** Is T Onto? Is T one-to-one? explain briefly

$Z(T) \neq$ origin $\therefore T$ is not 1 to 1.

$\text{Dim}(\text{Range}(T)) + \text{Dim}(Z(T)) = \text{Dim}(\text{Domain}(T))$

$\text{Dim}(\text{Range}(T)) = 4 - 2 = 2$

$\text{Dim}(\text{codomain}(T)) = 4$

$\text{Dim}(\text{Range}(T)) \neq \text{Dim}(\text{codomain}(T)) \therefore T$ is not onto.

✓

QUESTION 3. (10 points) Let $A = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$. If A is diagnolizable, then find an invertible matrix Q and a diagonal matrix D such that $QDQ^{-1} = A$, and hence $Q^{-1}AQ = D$.

$A = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$ $\quad C_A(\alpha) = |\alpha I_2 - A| = \begin{vmatrix} \alpha-2 & -4 \\ 1 & \alpha+3 \end{vmatrix} = (\alpha-2)(\alpha+3)+4$

$= \alpha^2 + \cancel{5}\alpha \cancel{+10} - 2$

$= (\alpha+2)(\alpha-1)$

$\therefore$ $\alpha = -2, 1$ are the eigenvalues

$\because$ Each α is only repeated once, A is diagnolizable.

$D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

~~$Q = E_{-2}$:~~ $\cancel{\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}}$ $\quad E_{-2}: \begin{array}{cc} a & b \\ \end{array}$ $\left[\begin{array}{cc|c} -4 & -4 & 0 \\ 1 & 1 & 0 \end{array}\right]$ $\quad a = -b$

$E_{-2} = \{(-b, b) \mid b \in \mathbb{R}\} = \text{span}\{(-1, 1)\}$

$E_1: \left[\begin{array}{cc|c} -1 & -4 & 0 \\ 1 & 4 & 0 \end{array}\right]$ $\quad a = -4b$

$E_1 = \{(-4b, b) \mid b \in \mathbb{R}\} = \text{span}\{(-4, 1)\}$

$Q = \begin{bmatrix} -1 & -4 \\ 1 & 1 \end{bmatrix}$

QUESTION 4. (8 points)

(a) Consider the normal dot product on $D = span\{(1,1,1),\ (-1,0,0)\}$. Use Gram-Schmidt algorithm and find an orthogonal basis for D.

$$D = span\{\overset{Q_1}{(1,1,1)}, \overset{Q_2}{(-1,0,0)}\}$$

$$O = \{w_1, w_2\}$$

$$w_1 = Q_1 = (1,1,1)$$

$$w_2 = Q_2 - \frac{<Q_2, w_1>}{|w_1|^2} w_1 = (-1,0,0) - \frac{-1}{1+1+1}(1,1,1)$$

$$= (-1,0,0) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$O = \left\{(1,1,1), \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$$

(b) Consider the integral inner product on $D = span\{1, x\}$, where $a = 0$ and $b = 1$. Use Gram-Schmidt algorithm and find an orthogonal basis for D.

$$D = span\{\overset{f_1}{1}, \overset{f_2}{x}\}$$

$$O = \{w_1, w_2\}$$

$$w_1 = f_1 = 1$$

$$w_2 = f_2 - \frac{<f_2, w_1>}{|w_1|^2} w_1 = x - \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx}(1)$$

$$= x - \frac{[x^2/2]_0^1}{[x]_0^1}$$

$$= x - \frac{1}{2}$$

$$O = \left\{1, x - \frac{1}{2}\right\}$$