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ON ϕ-PRÜFER RINGS AND ϕ-BEZOUT RINGS

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Abstract. The purpose of this paper is to introduce two new classes of rings that are closely related to the classes of Prüfer domains and Bezout domains. Let $\mathcal{H} = \{R \mid R$ is a commutative ring with $1 \neq 0$ and $Nil(R)$ is a divided prime ideal of $R\}$. Let $R \in \mathcal{H}$, $T(R)$ be the total quotient ring of R, and set $\phi : T(R) \longrightarrow R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. A nonnil ideal I of R is said to be ϕ-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. If every finitely generated nonnil ideal of R is ϕ-invertible, then we say that R is a *ϕ-Prüfer ring*. Also, we say that R is a *ϕ-Bezout ring* if $\phi(I)$ is a principal ideal of $\phi(R)$ for every finitely generated nonnil ideal I of R. We show that the theories of ϕ-Prüfer and ϕ-Bezout rings resemble that of Prüfer and Bezout domains.

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, $Nil(R)$ denotes the set of nilpotent elements of R, and $Z(R)$ denotes the set of zerodivisors of R. We start by recalling some background material. Recall that a non-zerodivisor of a ring R is called a *regular element* and an ideal of R is said to be *regular* if it contains a regular element. A ring R is called a *Prüfer ring*, in the sense of [13], if every finitely generated regular ideal of R is invertible, i.e., if I is a finitely generated regular ideal of R and $I^{-1} = \{x \in T(R) \mid xI \subset R\}$, then $II^{-1} = R$. A Prüfer domain is a Prüfer ring and a homomorphic image of a Prüfer domain is a Prüfer ring. Many characterizations and properties of Prüfer rings are stated in [13], [8], [9], [1], and [18]. For further study of Prüfer domains and Prüfer rings, we recommend [16], [12], [17], [14], and [11]. Recall from [9] that a ring R

is called a *pre-Prüfer ring* if every proper homomorphic image of R is a Prüfer ring, i.e., if R/I is a Prüfer ring for each nonzero proper ideal I of R. In [9], it was shown that the class of Prüfer rings and the class of pre-Prüfer rings are not comparable under set inclusion. A ring R is called a *Bezout ring*, in the sense of [14], if every finitely generated regular ideal of R is principal. A ring R is said to be a *chained ring* if for every $a, b \in R$, either $a \mid b$ or $b \mid a$ in R.

Recall from [10] and [7] that a prime ideal of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. In [2], [3], [4], [5], and [6], the second-named author investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$. In this paper, we give a generalization of Prüfer domains to the context of rings that are in the class $\mathcal{H}$. An ideal I of a ring R is said to be a *nonnil ideal* if $I \not\subset Nil(R)$. Recall from [2] that for a ring $R \in \mathcal{H}$ with total quotient ring $T(R)$, if $a \in R$ and $b \in R \setminus Z(R)$, then $\phi : T(R) \longrightarrow R_{Nil(R)}$ such that $\phi(a/b) = a/b$ is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. A nonnil ideal I of R is said to be *ϕ-invertible* if $\phi(I)$ is an invertible ideal of $\phi(R)$. If every finitely generated nonnil ideal of R is ϕ-invertible, then we say that R is a *ϕ-Prüfer ring*. Also, we say that R is a *ϕ-Bezout ring* if $\phi(I)$ is a principal ideal of $\phi(R)$ for every finitely generated nonnil ideal I of R. Recall from [4] that a ring $R \in \mathcal{H}$ is called a *ϕ-chained ring (ϕ-CR)* if $x^{-1} \in \phi(R)$ for every $x \in R_{Nil(R)} \setminus \phi(R)$; equivalently, if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ or $b \mid a$ in R. Clearly a chained ring is also a ϕ-chained ring. It was shown in [4] that for each integer $n \geq 1$, there is a ϕ-chained ring with Krull dimension n which is not a chained ring. Among many results in this paper, we show (Corollary 2.10) that a ring $R \in \mathcal{H}$ is a ϕ-Prüfer ring iff $\phi(R)$ is a Prüfer ring, iff R_P is a ϕ-CR for every prime ideal P of R, iff R_M is a ϕ-CR for every maximal ideal M of R, iff $R/Nil(R)$ is a Prüfer domain, iff $\phi(R)/Nil(\phi(R))$ is a Prüfer domain. Also, we show (Corollary 3.5) that a ring $R \in \mathcal{H}$ is a ϕ-Bezout ring iff $\phi(R)$ is a Bezout ring, iff $R/Nil(R)$ is a Bezout domain, iff $\phi(R)/Nil(\phi(R))$ is a Bezout domain, iff every finitely generated nonnil ideal of R is principal. A ϕ-Prüfer ring is a Prüfer ring and a ϕ-Bezout ring is a Bezout ring. We give an example (Example 2.15) of a Prüfer ring in $\mathcal{H}$ which is not a ϕ-Prüfer ring, and an example (Example 3.6) of a Bezout ring in $\mathcal{H}$ which is not a ϕ-Bezout ring.

Observe that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}$, $Ker(\phi) \subset Nil(R)$, $Nil(T(R)) = Nil(R)$, $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{Nil(R)}$ is quasilocal with maximal ideal $Nil(\phi(R))$, and $R_{Nil(R)}/Nil(\phi(R)) =$

$T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R))$. If I is a nonnil ideal of a ring $R \in \mathcal{H}$, then observe that $Nil(R) \subset I$, and if I is a nonnil finitely generated, then these generators can be chosen to be nonnilpotent elements of R. Also, if J is a finitely generated regular ideal of $\phi(R)$ for some $R \in \mathcal{H}$, then $Nil(\phi(R)) = Z(\phi(R)) \subset J$ and J can be generated by a finite number of regular elements of J, say, $\phi(x_1), \ldots, \phi(x_n)$ for some nonnilpotent elements x_i of R.

2. ϕ-Prüfer Rings

We start with the following lemma.

Lemma 2.1. *Let $R \in \mathcal{H}$ and let I be an ideal of R. Then I is a finitely generated nonnil ideal of R if and only if $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$.*

Proof. Suppose that I is a finitely generated nonnil ideal of R. Then it is clear that $\phi(I)$ is a finitely generated nonnil ideal of $\phi(R)$. Since $Nil(\phi(R)) = Z(\phi(R))$, we conclude that $\phi(I)$ is regular. Conversely, assume that $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$. Thus $\phi(I) = (\phi(x_1), \ldots, \phi(x_n))$ for some nonnilpotent elements $x_1, \ldots, x_n$ of I. Now, let y be a nonnilpotent element of I. Then $\phi(y) = \phi(c_1)\phi(x_1) + \cdots + \phi(c_n)\phi(x_n)$ for some elements c_j of R. Since $Ker(\phi) \subset Nil(R)$, we conclude that $y + d = c_1x_1 + \cdots + c_nx_n$ in R for some $d \in Nil(R)$. Hence $d = wy$ for some $w \in Nil(R)$ since $Nil(R)$ is a divided prime ideal. Thus $y+d = y(1+w) = c_1x_1+\cdots+c_nx_n$ in R. Since $1+w$ is a unit of R, we conclude that $y \in (x_1, \ldots, x_n)$, and hence $I = (x_1, \ldots, x_n)$ is a finitely generated nonnil ideal of R. □

The following is a characterization of ϕ-Prüfer rings in terms of Prüfer rings.

Theorem 2.2. *Let $R \in \mathcal{H}$. Then R is a ϕ-Prüfer ring if and only if $\phi(R)$ is a Prüfer ring.*

Proof. Suppose that R is a ϕ-Prüfer ring, and let J be a finitely generated regular ideal of $\phi(R)$. Since $J = \phi(I)$ for some ideal I of R and J is regular, we conclude that I is a nonnil finitely generated ideal of R by Lemma 2.1. Hence $J = \phi(I)$ is an invertible ideal of $\phi(R)$. Thus $\phi(R)$ is a Prüfer ring. Conversely, suppose that $\phi(R)$ is a Prüfer ring, and let I be a finitely generated nonnil ideal of R. Then $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$ by Lemma 2.1. Hence $\phi(I)$ is an invertible ideal of $\phi(R)$, and thus R is a ϕ-Prüfer ring. □

Before we give our next characterization of ϕ-Prüfer rings, we need the following three lemmas.

Lemma 2.3. *Let* $R \in \mathcal{H}$ *with* $Nil(R) = Z(R)$*, and let* I *be an ideal of* R*. Then* I *is an invertible ideal of* R *if and only if* $I/Nil(R)$ *is an invertible ideal of* $R/Nil(R)$*.*

PROOF. Suppose that I is an invertible ideal of R. Then I is a regular ideal of R, and thus $Nil(R) \subset I$ since $Nil(R)$ is a divided prime ideal of R. Hence $x_1i_1 + \cdots + x_ni_n = 1$ in R for some $x_j \in I^{-1}$ and $i_j \in I$. Since $Z(R) = Nil(R)$ is a divided prime ideal of R, $T(R)/Nil(R)$ is the quotient field of $R/Nil(R)$. Thus $x_1 + Nil(R), \ldots, x_n + Nil(R) \in (I/Nil(R))^{-1} \subset T(R)/Nil(R)$, and $(x_1i_1 + \cdots + x_ni_n) + Nil(R) = 1 + Nil(R)$ in $R/Nil(R)$. Hence $I/Nil(R)$ is an invertible ideal of $R/Nil(R)$. Conversely, suppose that $I/Nil(R)$ is an invertible ideal of $R/Nil(R)$. (Note that if $I \subset Nil(R)$, then $(I + Nil(R))/Nil(R) = 0$, and hence I is not invertible.) Once again, since $T(R)/Nil(R)$ is the quotient field of $R/Nil(R)$, it is easy to see that $(x_1i_1+\cdots+x_ni_n)+Nil(R) = 1+Nil(R)$ in $R/Nil(R)$ for some $x_j \in I^{-1}$ and $i_j \in I$. Thus $x_1i_1 + \cdots + x_ni_n = 1 + w$ in R for some $w \in Nil(R)$. Since $1 + w$ is a unit of R, we conclude that I is an invertible ideal of R. $\square$

Lemma 2.4. *Let* $R \in \mathcal{H}$ *and let* I *be an ideal of* R*. Then* I *is a finitely generated nonnil ideal of* R *if and only if* $I/Nil(R)$ *is a finitely generated nonzero ideal of* $R/Nil(R)$*.*

PROOF. Suppose that I is a finitely generated nonnil ideal of R. Then $Nil(R) \subset I$ since I is nonnil, and hence $I/Nil(R)$ is a finitely generated nonzero ideal of $R/Nil(R)$. Conversely, assume that $I/Nil(R) = (x_1 + Nil(R), \ldots, x_n + Nil(R))$ is a finitely generated nonzero ideal of $R/Nil(R)$ for some nonnilpotent elements $x_1, \ldots, x_n$ of I. Let y be a nonnilpotent element of I. Then $y + Nil(R) = (c_1x_1 + \cdots + c_nx_n) + Nil(R)$ in $R/Nil(R)$ for some $c_j \in R$. Thus $y + d = c_1x_1 + \cdots + c_nx_n$ in R for some $d \in Nil(R)$. Hence $d = wy$ for some $w \in Nil(R)$, and thus $y + d = y(1 + w) = c_1x_1 + \cdots + c_nx_n$ in R. Since $1 + w$ is a unit of R, we conclude that $y \in (x_1, \ldots, x_n)$, and hence $I = (x_1, \ldots, x_n)$ is a finitely generated nonnil ideal of R. $\square$

Lemma 2.5. *Let* $R \in \mathcal{H}$ *and let* P *be a prime ideal of* R*. Then* R/P *is ring-isomorphic to* $\phi(R)/\phi(P)$*.*

PROOF. Let $\alpha : R \longrightarrow \phi(R)/\phi(P)$ such that $\alpha(x) = \phi(x) + \phi(P)$. Then α is a ring homomorphism from R onto $\phi(R)/\phi(P)$ with $Ker(\alpha) = \phi^{-1}(\phi(P)) = P$ since $Ker(\phi) \subset Nil(R) \subset P$. Thus R/P is ring-isomorphic to $\phi(R)/\phi(P)$. $\square$

We next state the first main result of this paper.

Theorem 2.6. *Let $R \in \mathcal{H}$. Then R is a ϕ-Prüfer ring if and only if $R/Nil(R)$ is a Prüfer domain.*

Proof. Suppose that R is a ϕ-Prüfer ring, and let J be a finitely generated nonzero ideal of $\phi(R)/Nil(\phi(R))$. Then $J = \phi(I)/Nil(\phi(R))$ for some finitely generated nonnil ideal I of R by Lemma 2.4. Since $\phi(I)$ is an invertible ideal of $\phi(R) \in \mathcal{H}$ and $Nil(\phi(R)) = Z(\phi(R))$ is a divided prime ideal of $\phi(R)$, we conclude that $J = \phi(I)/Nil(\phi(R))$ is an invertible ideal of $\phi(R)/Nil(\phi(R))$ by Lemma 2.3. Hence $\phi(R)/Nil(\phi(R))$ is a Prüfer domain. Since $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$ by Lemma 2.5, we conclude that $R/Nil(R)$ is a Prüfer domain. Conversely, suppose that $R/Nil(R)$ is a Prüfer domain. Hence $\phi(R)/Nil(\phi(R))$ is a Prüfer domain by Lemma 2.5. Let I be a finitely generated nonnil ideal of R. Then $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$ by Lemma 2.1. Since $Nil(\phi(R)) = Z(\phi(R))$ is a divided prime ideal of $\phi(R)$ and $\phi(I)/Nil(\phi(R))$ is an invertible ideal of $\phi(R)/Nil(\phi(R))$ by Lemma 2.4, we conclude that $\phi(I)$ is an invertible ideal of $\phi(R) \in \mathcal{H}$ by Lemma 2.3, and thus R is a ϕ-Prüfer ring. $\square$

Recall from [4] that a ring $R \in \mathcal{H}$ is called a *ϕ-chained ring (ϕ-CR)* if for every $x \in R_{Nil(R)} \setminus \phi(R)$, we have $x^{-1} \in \phi(R)$; equivalently, if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ or $b \mid a$ in R. The following is a characterization of ϕ-CR's in terms of valuation domains.

Theorem 2.7. *Let $R \in \mathcal{H}$. Then R is a ϕ-CR if and only if $R/Nil(R)$ is a valuation domain.*

Proof. Set $D = R/Nil(R)$. Suppose that R is a ϕ-CR. Let $x = a + Nil(R)$, $y = b + Nil(R) \in D$, where $a, b \in R \setminus Nil(R)$. Since either $a \mid b$ or $b \mid a$ in R, we conclude that either $x \mid y$ or $y \mid x$ in D. Hence D is a valuation domain. Conversely, suppose that D is a valuation domain, and let $a, b \in R \setminus Nil(R)$. Then $x = a + Nil(R)$, $y = b + Nil(R)$ are nonzero elements of D. Hence $x \mid y$ or $y \mid x$ in D; we may assume that $x \mid y$ in D. Thus $b = ad + w$ in R for some $d \in R$ and $w \in Nil(R)$. Since $Nil(R) \subset (a)$, we have $w = as$ for some $s \in Nil(R)$. Thus $b = ad + w = a(d + s)$ in R, and hence $a \mid b$ in R. Thus R is a ϕ-CR. $\square$

Corollary 2.8. *Let $R \in \mathcal{H}$ be a ϕ-CR. Then R is a ϕ-Prüfer ring.*

Proof. By Theorem 2.7, we have that $R/Nil(R)$ is a valuation domain, and hence is a Prüfer domain. Thus R is a ϕ-Prüfer ring by Theorem 2.6. $\square$

It is well-known [16, Theorem 64] that an integral domain R is a Prüfer domain iff R_P is a valuation domain for each prime ideal P of R, iff R_M is a valuation domain for each maximal ideal M of R. The following is the analogous characterization of ϕ-Prüfer rings in terms of ϕ-CR's.

Theorem 2.9. *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(1) *R is a ϕ-Prüfer ring;*
(2) *R_P is a ϕ-CR for each prime ideal P of R;*
(3) *R_M is a ϕ-CR for each maximal ideal M of R.*

Proof. Set $D = R/Nil(R)$. **(1) $\Longrightarrow$ (2)**. Since D is a Prüfer domain by Theorem 2.6, we conclude that $D_{P/Nil(R)}$ is a valuation domain for each prime ideal P of R by [16, Theorem 64]. Since $D_{P/Nil(R)}$ is ring-isomorphic to $R_P/Nil(R)R_P = R_P/Nil(R_P)$ and $R_P \in \mathcal{H}$, we conclude that R_P is a ϕ-CR by Theorem 2.7. **(2) $\Longrightarrow$ (3)**. Clear. **(3) $\Longrightarrow$ (1)**. Since $R_M \in \mathcal{H}$ for each maximal ideal M of R, we conclude that $R_M/Nil(R_M)$ is evaluation domain for each maximal ideal M of R by Theorem 2.7. Hence $D_{M/Nil(R)}$ is a valuation domain for each maximal ideal M of R. Thus $R/Nil(R)$ is a Prüfer domain by [16, Theorem 64], and hence R is a ϕ-Prüfer ring by Theorem 2.6. $\square$

Combining Theorem 2.2, Lemma 2.5, and Theorems 2.6, 2.7, and 2.9, we arrive at the following corollary.

Corollary 2.10. *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(1) *R is a ϕ-Prüfer ring;*
(2) *$\phi(R)$ is a Prüfer ring;*
(3) *$R/Nil(R)$ is a Prüfer domain;*
(4) *$\phi(R)/Nil(\phi(R))$ is a Prüfer domain;*
(5) *R_P is a ϕ-CR for each prime ideal P of R;*
(6) *$R_P/Nil(R_P)$ is a valuation domain for each prime ideal P of R;*
(7) *$R_M/Nil(R_M)$ is a valuation domain for each maximal ideal M of R;*
(8) *R_M is a ϕ-CR for each maximal ideal M of R.* $\square$

It is well-known ([16, Theorem 65]) that a valuation overring of a Prüfer domain R is of the form R_P for some prime ideal P of R. We have a similar result for ϕ-Prüfer rings. Recall that an overring of a ring R is a ring between R and $T(R)$.

Theorem 2.11. *Let $R \in \mathcal{H}$ be a ϕ-Prüfer ring and let S be an a ϕ-chained overring of R. Then $S = R_P$ for some prime ideal P of R containing $Z(R)$.*

PROOF. Since $Nil(R) = Nil(S)$, $S \in \mathcal{H}$. Since S is quasilocal by [4], let M be the maximal ideal of S. Then $M \cap R = P$ is a prime ideal of R. Since S is quasilocal, P must contain $Z(R)$. Now, we may consider $S/Nil(R)$ as an overring of $D = R/Nil(R)$. Since D is a Prüfer domain by Theorem 2.6 and $S/Nil(R)$ is a valuation domain by Theorem 2.7, we have $S/Nil(R) = D_{P/Nil(R)} = R_P/Nil(R)$ by [16, the proof of Theorem 65], and hence $S = R_P$. □

It is well-known ([16, Exercise 13, page 42]) that a finitely generated nonzero prime ideal of a Prüfer domain R is maximal. We have a similar result for ϕ-Prüfer rings.

Theorem 2.12. *Let $R \in \mathcal{H}$ be a ϕ-Prüfer ring. If P is a finitely generated nonnil prime ideal of R, then P is a maximal ideal of R.*

PROOF. Set $D = R/Nil(R)$. Then D is a Prüfer domain by Theorem 2.6. Since $P/Nil(R)$ is a finitely generated nonzero ideal of D, we have that $P/Nil(R)$ is a maximal ideal of D by [16, Exercise 13, page 42], and hence P is a maximal ideal of R. □

Recall ([15] or [17, Exercise 18, page 150]) that a ring R is called an *arithmetical ring* if R_M is a chained ring for every maximal ideal M of R. Since a chained ring is a ϕ-chained ring, we conclude that if $R \in \mathcal{H}$ is an arithmetical ring, then R is a ϕ-Prüfer ring by Theorem 2.9. Since a ϕ-chained ring need not be a chained ring by [4], a ϕ-chained ring is quasilocal by [4], and a ϕ-chained ring is a ϕ-Prüfer ring by Corollary 2.8, we conclude that a ϕ-Prüfer ring need not be an arithmetical ring.

We will next prove that a ϕ-Prüfer ring is a Prüfer ring; but first we need a lemma.

Lemma 2.13. *Let $R \in \mathcal{H}$ and $x \in T(R)$. If $\phi(x) \in \phi(R)$, then $x \in R$. In particular, if $\phi(R)$ is integrally closed in $T(\phi(R)) = R_{Nil(R)}$, then R is integrally closed in $T(R)$.*

PROOF. Suppose that $\phi(x) \in \phi(R)$. We may assume that $x \notin Nil(R)$. Hence $\phi(x) = \phi(s)$ for some nonnilpotent $s \in R$. Thus $w = x - s \in Ker(\phi) \subset Nil(R)$, and hence $x = s + w \in R$. Suppose that $\phi(R)$ is integrally closed in $T(\phi(R))$ and $x \in T(R)$ is integral over R. Once again, we may assume that $x \notin Nil(R)$. Since $\phi(x) \in T(\phi(R))$, it is easy to see that $\phi(x)$ is integral over $\phi(R)$. Thus $\phi(x) \in \phi(R)$, and hence $x \in R$. Thus R is integrally closed in $T(R)$. □

Theorem 2.14. *Let $R \in \mathcal{H}$. If R is a ϕ-Prüfer ring, then R is a Prüfer ring.*

Proof. Suppose that R is a ϕ-Prüfer ring. Then $\phi(R)$ is a Prüfer ring by Theorem 2.2. Hence every overring of $\phi(R)$ is integrally closed in $T(\phi(R)) = R_{Nil(R)}$ by [14, Theorem 6.2]. Let S be an overring of R. Then $Nil(R) = Nil(S)$, and therefore $S \in \mathcal{H}$. Since $\phi(S)$ is an overring of $\phi(R)$, $\phi(S)$ is integrally closed in $T(\phi(S))$, and hence S is integrally closed in $T(S)$ by Lemma 2.13. Thus R is a Prüfer ring by [14, Theorem 6.2]. □

If R is a Prüfer ring and $R \notin \mathcal{H}$, then R is not a ϕ-Prüfer ring by definition. The following example shows that for each integer $n \geq 1$, there is a Prüfer ring $R \in \mathcal{H}$ with Krull dimension n which is not a ϕ-Prüfer ring. Our example relies on the idealization construction as in [14, Chapter VI, page 161].

Example 2.15. *Let $n \geq 1$ be an integer and let D be a non-integrally closed domain with Krull dimension n and quotient field L. Set $R = D(+)(L/D)$. Then $R \in \mathcal{H}$ is a Prüfer ring with Krull dimension n which is not a ϕ-Prüfer ring.*

Proof. By [14, Theorem 25.1(3)], R has Krull dimension n. Now, $Nil(R) = \{0\}(+)(L/D)$ is a divided prime ideal of R. For let $(0, y + D) \in Nil(R)$ and $(a, x + D) \in R \setminus Nil(R)$; then $(0, y + D) = (a, x + D)(0, y/a + D)$. Thus $R \in \mathcal{H}$. Since every nonunit of R is a zerodivisor, we conclude that R is a Prüfer ring. Since D is a non-integrally closed domain and $R/Nil(R)$ is ring-isomorphic to D, we conclude that $R/Nil(R)$ is not a Prüfer domain, and hence R is not a ϕ-Prüfer ring by Theorem 2.6. □

In view of Theorem 2.14 and Example 2.15, we have the following result.

Theorem 2.16. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a Prüfer ring if and only if R is a ϕ-Prüfer ring.*

Proof. Suppose that R is a Prüfer ring. Then $\phi(R) = R$ is a Prüfer ring, and hence R is a ϕ-Prüfer ring by Theorem 2.2. The converse is clear by Theorem 2.14. □

Observe that if every overring of $R \in \mathcal{H}$ is integrally closed, then R need not be a ϕ-Prüfer ring by Example 2.15. However, we have the following result.

Theorem 2.17. *Suppose that $R \in \mathcal{H}$. Then R is a ϕ-Prüfer ring if and only if every overring of $\phi(R)$ is integrally closed.*

PROOF. Since R is a ϕ-Prüfer ring iff $\phi(R)$ is a Prüfer ring by Theorem 2.2 and $\phi(R)$ is a Prüfer ring iff every overring of $\phi(R)$ is integrally closed by [14, Theorem 6.2], the claim is now clear. $\square$

In the following example, we will show that for each integer $n \geq 1$, there is a (non-domain) ϕ-Prüfer ring with Krull dimension n.

Example 2.18. *Let $n \geq 1$ be an integer and let D be a Prüfer domain with Krull dimension n and quotient field L. Then $R = D(+)L \in \mathcal{H}$ is a (non-domain) ϕ-Prüfer ring with Krull dimension n.*

PROOF. Once again, by [14, Theorem 25.1(3)] R has Krull dimension n. Also, $Nil(R) = \{0\}(+)L$ is a divided prime ideal of R. For let $(0, x) \in Nil(R)$ and $(a, y) \in R \setminus Nil(R)$; then $(0, x) = (a, y)(0, x/a)$. Hence $R \in \mathcal{H}$. Since $R/Nil(R)$ is ring-isomorphic to D and D is a Prüfer domain, we conclude that R is a ϕ-Prüfer ring by Theorem 2.6. $\square$

Recall from [9] that a ring R is called a *pre-Prüfer ring* if every proper homomorphic image of R is a Prüfer ring, i.e., if R/I is a Prüfer ring for every nonzero proper ideal I of R.

Theorem 2.19. *Let $R \in \mathcal{H}$ with $Nil(R) \neq \{0\}$. Then R is a pre-Prüfer ring if and only if R is a ϕ-Prüfer ring.*

PROOF. Suppose that R is a pre-Prüfer ring. Since $Nil(R) \neq \{0\}$, $R/Nil(R)$ is a Prüfer ring (domain). Hence R is a ϕ-Prüfer ring by Theorem 2.6. Conversely, suppose that R is a ϕ-Prüfer ring, and let I be a nonzero proper ideal of R. Then either $I \subset Nil(R)$ or $Nil(R) \subset I$. Suppose that $I \subset Nil(R)$. Set $D = R/I$. Since $Nil(D) = Nil(R)/I$ is a divided prime ideal of D, $D \in \mathcal{H}$. Since $D/Nil(D)$ is ring-isomorphic to $R/Nil(R)$ and $R/Nil(R)$ is a Prüfer domain, we conclude that D is a ϕ-Prüfer ring by Theorem 2.6. Hence D is a Prüfer ring by Theorem 2.14. Now, assume that $Nil(R) \subset I$. Let $J = I/Nil(R)$. Since $S = R/Nil(R)$ is a Prüfer domain by Theorem 2.6 and a homomorphic image of a Prüfer domain is a Prüfer ring, we conclude that S/J is Prüfer ring. Since S/J is ring-isomorphic to R/I, we conclude that R/I is a Prüfer ring, and hence R is a pre-Prüfer ring. $\square$

Observe that if $R \in \mathcal{H}$ and $Nil(R) = \{0\}$, then R is an integral domain. The following example shows that the hypothesis $Nil(R) \neq \{0\}$ in Theorem 2.19 is crucial.

Example 2.20. *(*[18, Example 2.9]*) Let D be a Prüfer domain with quotient field F. For indeterminates X and Y, let $K = F(Y)$ and let V be the valuation domain $K + XK[[X]]$. Then V is one-dimensional with maximal ideal $M = XK[[X]]$. The ring $R = D + M$ is a pre-Prüfer ring (domain) which is not a Prüfer ring(domain). Hence R is not a ϕ-Prüfer ring.* □

3. ϕ-Bezou Rings

Once again, throughout this section $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$. Recall that a ring R is called a *Bezout ring* if every finitely generated regular ideal of R is principal. We say that $R \in \mathcal{H}$ is a *ϕ-Bezout ring* if $\phi(I)$ is a principal ideal of $\phi(R)$ for every finitely generated nonnil ideal I of R; equivalently, if $\phi(I)$ is a principal ideal of $\phi(R)$ for every 2-generated nonnil ideal I of R. It is clear that a ϕ-Bezout ring is a ϕ-Prüfer ring. Since a Prüfer domain need not be a Bezout domain, a ϕ-Prüfer ring need not be a ϕ-Bezout ring. We start with the following lemma.

Lemma 3.1. *Let $R \in \mathcal{H}$ and let I be an ideal of R. Then I is a principal nonnil ideal of R if and only if $I/Nil(R)$ is a nonzero principal ideal of $R/Nil(R)$.*

Proof. The proof is similar to the proof of Lemma 2.4, and hence we leave the proof to the reader. □

Theorem 3.2. *Let $R \in \mathcal{H}$. Then R is a ϕ-Bezout ring if and only if every finitely generated nonnil ideal of R is principal. In particular, if R is a ϕ-Bezout ring, then R is a Bezout ring.*

Proof. Suppose that R is a ϕ-Bezout ring, and let I be a finitely generated nonnil ideal of R. Hence $\phi(I)$ is principal. Since $\phi(R) \in \mathcal{H}$, $\phi(I)/Nil(\phi(R))$ is a principal ideal of $\phi(R)/Nil(\phi(R)$. Since $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R)$, $I/Nil(R)$ is a principal ideal of $R/Nil(R)$, and thus I is principal by Lemma 3.1. Conversely, suppose that every finitely generated nonnil ideal of R is principal, and let I be a finitely generated nonnil ideal of R. Then $\phi(I)$ is a principal ideal of $\phi(R)$, and hence R is a ϕ-Bezout ring. The "in particular" statement is clear. □

In the following result, we give a characterization of ϕ-Bezout rings in terms of Bezout domains.

Theorem 3.3. *Let $R \in \mathcal{H}$. Then R is a ϕ-Bezout ring if and only if $R/Nil(R)$ is a Bezout domain.*

PROOF. Set $D = R/Nil(R)$. Suppose that R is a ϕ-Bezout ring, and let J be a finitely generated nonzero ideal of D. Then $J = I/Nil(R)$ for some finitely generated nonnil ideal I of R by Lemma 2.4. Since I is principal, we conclude that J is principal by Lemma 3.1, and thus D is a Bezout domain. Conversely, suppose that D is a Bezout domain, and let I be a finitely generated nonnil ideal of R. Hence $I/Nil(R)$ is a nonzero principal ideal of D, and thus I is a principal ideal of R by Lemma 3.1. Hence R is a ϕ-Bezout ring by Theorem 3.2. ☐

The following theorem is a characterization of ϕ-Bezout rings in terms of Bezout rings.

Theorem 3.4. *Let $R \in \mathcal{H}$. Then R is a ϕ-Bezout ring if and only if $\phi(R)$ is a Bezout ring.*

PROOF. Suppose that R is a ϕ-Bezout ring. Then $R/Nil(R)$ is a Bezout domain by Theorem 3.3. Let J be a finitely generated regular ideal of $\phi(R)$. Since $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$ by Lemma 2.5, we conclude that $J/Nil(\phi(R))$ is a nonzero principal ideal of $\phi(R)/Nil(\phi(R))$, and hence J is a principal ideal of $\phi(R)$ by Lemma 3.1. Thus $\phi(R)$ is a Bezout ring. Conversely, suppose that $\phi(R)$ is a Bezout ring, and let I be a finitely generated nonnil ideal of R. Then $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$, and thus $\phi(I)$ is principal. Hence R is a ϕ-Bezout ring. ☐

Combining Theorems 3.2, 3.3, and 3.4, we arrive at the following corollary.

Corollary 3.5. *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(1) *R is a ϕ-Bezout ring;*
(2) *$\phi(R)$ is a Bezout ring;*
(3) *$\phi(R)/Nil(\phi(R))$ is a Bezout domain;*
(4) *$R/Nil(R)$ is a Bezout domain;*
(5) *Every finitely generated nonnil ideal of R is principal.* ☐

It is clear by Theorem 3.2 that if R is a ϕ-Bezout ring, then R is a Bezout ring. The following is an example of a Bezout ring $R \in \mathcal{H}$ which is not a ϕ-Bezout ring.

Example 3.6. *Let $n \geq 1$ be an integer and let D be a non-Bezout domain with Krull dimension n and quotient field L. Then by a similar proof as in Example 2.15, $R = D(+)(L/D) \in \mathcal{H}$ is a Bezout ring with Krull dimension n which is not a ϕ-Bezout ring.* ☐

In view of Example 3.6, we have the following result.

Proposition 3.7. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-Bezout ring if and only if R is a Bezout ring.*

Proof. Just observe that in this case, we have $\phi(R) = R$. □

Example 3.8. *Let $n \geq 1$ be an integer and let D be a Bezout domain with Krull dimension n and quotient field L. Then $R = D(+)L \in \mathcal{H}$ is a (non-domain) ϕ-Bezout ring with Krull dimension n.*

Proof. By a similar argument as in the proof of Example 2.18, $R \in \mathcal{H}$ has Krull dimension n. Since $R/Nil(R)$ is ring-isomorphic to D, $R/Nil(R)$ is a Bezout domain, and hence R is a ϕ-Bezout ring by Theorem 3.3. □

It is well-known ([16, Theorem 63]) that a quasilocal domain is a valuation domain iff it is a Bezout domain. We have a similar result for ϕ-Bezout rings.

Theorem 3.9. *Let $R \in \mathcal{H}$ be a quasilocal ring. Then R is a ϕ-chained ring if and only if R is a ϕ-Bezout ring.*

Proof. Suppose that R is a ϕ-chained ring. Then R is a ϕ-Bezout ring by Theorem 3.2. Conversely, suppose that R is a ϕ-Bezout ring. Then $D = R/Nil(R)$ is a Bezout domain by Theorem 3.3. Since D is a quasilocal Bezout domain, D is a valuation domain by [16, Theorem 63], and hence R is a ϕ-chained ring by Theorem 2.7. □

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