A COUNTER EXAMPLE FOR A QUESTION ON PSEUDO-VALUATION RINGS

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Abstract. In this paper, we give a counter example of the following question which was raised by Anderson, Dobbs, and the author in [3, Question 3.14]: Let G be a strongly prime ideal of a ring D such that $G \subset Z(D)$ and $(G : G) = T(D)$ is a PVR. Then $T(D)$ has maximal ideal $Z(D)_S$, where $S = D \setminus Z(D)$, and $Z(D)$ is a prime ideal of D. Is $Z(D)$ also a strongly prime ideal of D?

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. The following notation will be used throughout. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, Nil(R) denotes the set of nilpotent elements of R, $Z(R)$ denotes the set of zerodivisors of R, $S = R \setminus Z(R)$, $dim(R)$ denotes the Krull dimension of R, and if B is an R-module, then $Z(B)$ denotes the set of zerodivisors on B, that is, $Z(B) = \{x \in R \mid xy = 0 \text{ in B for some } y \neq 0 \text{ and } y \in B\}$. If I is an ideal of R, then $(I : I) = \{x \in T(R) \mid xI \subset I\}$. We begin by recalling some background material. As in [20], an integral domain R, with quotient field K, is called a *pseudo-valuation domain (PVD)* in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. In [5], Anderson, Dobbs and the author generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [5] that a prime ideal P of R is said to be *strongly prime (in R)* if aP and bR are comparable (under inclusion) for all $a, b \in R$. A ring R is called a *pseudo-valuation ring (PVR)* if each prime ideal of R is strongly prime. A PVR is necessarily quasilocal [5, Lemma 1(b)]; a chained ring is a PVR [[5], Corollary 4]; and an integral domain is a PVR if and only if it is a PVD (cf. [1, Proposition 3.1], [2, Proposition 4.2], and [12, Proposition 3]). Recall from [13] and [17] that a prime ideal P of R is called *divided* if it is comparable (under inclusion) to every ideal of R. A ring R is called a *divided ring* if every prime ideal of R is divided. In [8], the author gives another generalization of PVDs to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [8] that for a ring R with total quotient ring $T(R)$ such that $Nil(R)$ is a divided prime ideal of R, let $\phi : T(R) \longrightarrow K := R_{Ni(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = x/1$ for every $x \in R$. A prime ideal Q of $\phi(R)$ is called a *K-strongly prime* if $xy \in Q$, $x \in K, y \in K$ implies that either $x \in Q$ or $y \in Q$. If each prime ideal of $\phi(R)$ is K-strongly prime, then $\phi(R)$ is called a *K-pseudo-valuation ring (K-PVR)*. A prime ideal P of R is called a *ϕ-strongly prime* if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a *ϕ-pseudo-valuation ring ($\phi - PVR$)*. It is shown in [8, Corollary 7(2)] that a ring

R is a ϕ-PVR if and only if $Nil(R)$ is a divided prime ideal of R and for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid ac$ in R for each nonunit $c \in R$. Since a PVR is a ϕ-PVR, it is shown in [9, Theorem 2.6] that for each $n \geq 0$ there is a ϕ-PVR with Krull dimension n which is not a PVR. For other related study on ϕ-rings, we recommend [10], [11], [6], [7], [14].

In this paper, we give a counter example of the following question that was raised by Anderson, Dobbs, and the author in [3, Question 3.14]: Let G be a strongly prime ideal of a ring D such that $G \subset Z(D)$ and $(G : G) = T(D)$ is a PVR. Then $T(D)$ has maximal ideal $Z(D)_S$, where $S = D \setminus Z(D)$, and $Z(D)$ is a prime ideal of D. Is $Z(D)$ also a strongly prime ideal of D?

Our counter example relies on the the idealization construction $R(+)B$ arising from a ring R and an R-module B as in Huckaba [21, Chapter VI]. We recall this construction. For a ring R, let B be an R-module. Consider $R(+)B = \{(r, b) : r \in R$, and $b \in B\}$, and let (r, b) and (s, c) be two elements of $R(+)B$. Define :

(1) $(r, b) = (s, c)$ if $r = s$ and $b = c$.
(2) $(r, b) + (s, c) = (r + s, b + c)$.
(3) $(r, b)(s, c) = (rs, bs + rc)$.

Under these definitions $R(+)B$ becomes a commutative ring with identity. In the following proposition, we state some basic properties of $R(+)B$.

PROPOSITION 1.1. *Let R be a ring, B be an R-module, and $Z(B)$ be the set of zerodivisors on B. Then:*

(1) *The ideal J of $R(+)B$ is prime (maximal) if and only if $J = P(+)B$, where P is a prime (maximal) ideal of R. Hence $dim(R) = dim(R(+)B)$* [21, Theorem 25.1].
(2) $(r, b) \in Z(R(+)B)$ *if and only if* $r \in Z(R) \cup Z(B)$ [21, Theorem 25.3].
(3) *If P is a prime ideal of R, then $(R(+)B)_{P(+)B}$ is ring-isomorphic to $R_P(+)B_P$* [21, Corollary 25.5(2)].

2. Counter example

Recall that if B is an R-module, then $Z(B) = \{x \in R \mid xy = 0$ in B for some $y \neq 0$ and $y \in B\}$. Also, recall that if R is an integral domain and B is an R-module, then B is said to be *divisible* if r is a nonzero element of R and $b \in B$, then there exists $f \in B$ such that $rf = b$. We start this section with the following lemma.

LEMMA 2.1. *Let R be an integral domain with quotient field F, P be a prime ideal of R, and $N = R \setminus P$. Then $B = F/P_N$ is a divisible R-module and $Z(B) = P$.*

Proof. It is clear that B is an R-module and $P \subset Z(B)$. Now, suppose that $x(y + F/P_N) = 0$ in B for some $x \in R \setminus P$. Hence $xy = p/n \in P_N$ for some $p \in P$ and $n \in N$. Thus $y = p/nx \in P_N$. Hence $y + F/P_N = 0$ in B. Thus $x \notin Z(B)$. Hence $Z(B) = P$. Next, we show that B is divisible. Let r be a nonzero element of R and $b = x + F/P_N \in B$. Then choose $f = x/r + F/P_N$. Hence $rf = b$, and thus B is divisible. □

The following three propositions are needed.

PROPOSITION 2.2. *Let V be a valuation domain of the form $F + M$, where F is a field and M is the maximal ideal of V, and let $R = D + M$ for some subring D of F.*

(1) *(*[16]*.) If P is a prime ideal of D, then $R_{P+M} = D_P + M$.*
(2) *(*[18, Proposition 4.9(i)]*.) R is a PVD if and only if either D is a PVD with quotient field F or D is a field.*

PROPOSITION 2.3. *(*[15, Theorem 3.1]*.) Let R be a ring and B be an R-module. Set $D = R(+)B$. Then:*

(1) *If D is a PVR, then R is a PVR.*
(2) *If R is a PVD and B is a divisible R-module, then $D = R(+)B$ is a PVR.*

Recall that an integral domain is called a *valuation* domain if for every $a, b \in R$, either $a \mid b$ in R or $b \mid a$ in R.

PROPOSITION 2.4. (1) *A valuation domain is a PVD (*[20, Proposition 1.1]*).*
(2) *A PVR is quasilocal (*[5, Lemma 1(b)]*).*
(3) *Let R be a ring. Then R is a PVR if and only if a maximal ideal of R is a strongly prime ideal (*[5, Theorem 2]*).*

Now, we state our example

EXAMPLE 2.5. *Let $\mathcal{Z}$ be the ring of integers with quotient field $\mathcal{Q}$. Let $R = \mathcal{Z} + X\mathcal{Q}[[X]]$, F be the quotient field of R, $P = 3\mathcal{Z} + X\mathcal{Q}[[X]]$ is a maximal ideal of R, $N = R \setminus P$, $B = F/P_N$ is an R-module, and set $D = R(+)B$. Then $Z(D) = P(+)B$ is a maximal ideal of D which is not a strongly prime ideal and $G = X\mathcal{Q}[[X]](+)B$ is a strongly prime ideal of D such that $G \subset Z(D)$ and $(G : G) = T(D)$ is a PVR.*

Proof. By Lemma 2.1 and Proposition 1.1(2), we conclude that $Z(D) = P(+)B$. By Proposition 1.1(1), $Z(D) = P(+)B$ is a maximal ideal of D. Since D is not quasilocal and $Z(D)$ is a maximal ideal of D, $Z(R)$ is not a strongly prime ideal of D by Proposition 2.4(2 and 3) . Now, $T(D)$ is ring-isomorphic to $R_P(+)B_P$ by Proposition 1.1(3). Since $R_P = \mathcal{Z}_{3\mathcal{Z}} + X\mathcal{Q}[[X]]$ by Proposition 2.2(1) and $B_P = B$ by the construction of B, we conclude that $T(D)$ is ring-isomorphic to $\mathcal{Z}_{3\mathcal{Z}} + X\mathcal{Q}[[X]](+)B$. Since it is well-known that $\mathcal{Z}_{3\mathcal{Z}} + X\mathcal{Q}[[X]]$ is a valuation domain and hence is a PVD by Proposition 3.4(1) and B is divisible by Lemma 2.1, we conclude that $\mathcal{Z}_{3\mathcal{Z}} + X\mathcal{Q}[[X]](+)B$ is a PVR by Proposition 2.3(2). Hence, $T(D)$ is a PVR and $G = X\mathcal{Q}[[X]](+)B$ is a strongly prime ideal of D. It is clear that $G \subset Z(D)$. Since $yX\mathcal{Q}[[X]] \subset X\mathcal{Q}[[X]]$ for every $y \in \mathcal{Z}_{3\mathcal{Z}} + X\mathcal{Q}[[X]]$, we have $(G : G) = T(D)$ is a PVR. □

Let R be a ring. Observe that if $Z(R)$ is a strongly prime ideal of R, then $(Z(R) : Z(R)) = T(R)$ is a PVR with maximal ideal $Z(R)$ by [3, Theorem 3.11(b)]. However, if G is a strongly prime ideal of R which is properly contained in $Z(R)$, then $(G : G) = T(R)$ need not be a PVR as in the following example.

EXAMPLE 2.6. *Let $\mathcal{Z}$ be the ring of integers and let $\mathcal{C}$ be the field of complex numbers. Let $R = \mathcal{Z} + X\mathcal{C}[[X]]$, F be the quotient field of R, $P = 3\mathcal{Z} + X\mathcal{C}[[X]]$ is a maximal ideal of R, $N = R \setminus P$, $B = F/P_N$ is an R-module, and set $D = R(+)B$. Then $Z(D) = P(+)B$ is a maximal ideal of D which is not a strongly prime ideal and $G = X\mathcal{C}[[X]](+)B$ is a strongly prime ideal of D such that $G \subset Z(D)$ and $(G : G) = T(D)$ is not a PVR.*

Proof. By an argument similar to that one just given in the proof of the above Example, we conclude that $Z(D) = P(+)X\mathcal{C}[[X]]$ and $T(D)$ is ring-isomorphic to

$L = \mathcal{Z}_{3\mathcal{Z}} + X\mathcal{C}[[X]](+)B$. Since $\mathcal{Z}_{3\mathcal{Z}} + X\mathcal{C}[[X]]$ is not a PVD by Proposition 2.2(2), we conclude that L is not a PVR by Proposition 2.3(1). Thus $T(D)$ is not a PVR. Now, since $T(D)$ is ring-isomorphic to L and $X\mathcal{C}[[X]]$ is a strongly prime ideal of R, G is a strongly prime ideal of D. $\square$

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