Communications in Algebra®, 35: 1167–1181, 2007

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870601141951

ON PSEUDO-ALMOST VALUATION DOMAINS

Ayman Badawi
Department of Mathematics and Statistics, American University of Sharjah, Sharjah, United Arab Emirates

Let R be an integral domain with quotient field K and integral closure R'. Anderson and Zafrullah called R an "almost valuation domain" if for every nonzero $x \in K$, there is a positive integer n such that either $x^n \in R$ or $x^{-n} \in R$. In this article, we introduce a new closely related class of integral domains. We define a prime ideal P of R to be a "pseudo-strongly prime ideal" if, whenever $x, y \in K$ and $xyP \subseteq P$, then there is a positive integer $m \geq 1$ such that either $x^m \in R$ or $y^m P \subseteq P$. If each prime ideal of R is a pseudo-strongly prime ideal, then R is called a "pseudo-almost valuation domain" (PAVD). We show that the class of valuation domains, the class of pseudo-valuation domains, the class of almost valuation domains, and the class of almost pseudo-valuation domains are properly contained in the class of pseudo-almost valuation domains; also we show that the class of pseudo-almost valuation domains is properly contained in the class of quasilocal domains with linearly ordered prime ideals. Among the properties of PAVDs, we show that an integral domain R is a PAVD if and only if for every nonzero $x \in K$, there is a positive integer $n \geq 1$ such that either $x^n \in R$ or $ax^{-n} \in R$ for every nonunit $a \in R$. We show that pseudo-almost valuation domains are precisely the pullbacks of almost valuation domains, we characterize pseudo-almost valuation domains of the form $D + M$, and we use this characterization to construct PAVDs that are not almost valuation domains. We show that if R is a Noetherian PAVD, then R has Krull dimension at most one and R' is a valuation domain; we show that every overring of a PAVD R is a PAVD iff R' is a valuation domain and every integral overring of R is a PAVD.

Key Words: Dedekind; Prime ideal; Prufer; Radical ideal; Valuation.

Mathematics Subject Classification: Primary 13A15; Secondary 13A10, 13F05.

1. INTRODUCTION

Throughout this article, R denotes an integral domain with quotient field K and integral closure R'. If I is an ideal of R, then $(I : I) = \{x \in K \mid xI \subseteq I\}$. We start by recalling some background material. In Hedstrom and Houston (1978a), the author introduced a class of integral domains which is closely related to the class of valuation domains (recall that an integral domain R is said to be a *valuation domain* if for every nonzero $x \in K$, either $x \in R$ or $x^{-1} \in R$). We recall from Hedstrom and Houston

Received November 14, 2005; Revised May 29, 2006. Communicated by I. Swanson.
Address correspondence to Ayman Badawi, Department of Mathematics and Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates; E-mail: abadawi@aus.edu

(1978a) that R is called a *pseudo-valuation domain* in case each prime ideal P of R is a *strongly prime ideal*, in the sense that $xy \in P$, $x, y \in K$ implies that either $x \in P$ or $y \in P$. It was shown in Hedstrom and Houston (1978a, Theorem 1.5(3)) that an integral domain R is a pseudo-valuation domain if and only if for every nonzero $x \in K$, either $x \in R$ or $ax^{-1} \in R$ for every nonunit $a \in R$. It is clear that a valuation domain is a pseudo-valuation domain. Also, it was shown in Hedstrom and Houston (1978b, Example 3.1) that for each $n \geq 1$, there is a pseudo-valuation domain with Krull dimension n which is not a valuation domain. In Badawi (2002), Houston and the author gave a generalization of pseudo-valuation domains: Recall from Badawi (2002) that R is said to be an *almost pseudo-valuation domain* in case each prime ideal P of R is a *strongly primary ideal*, in the sense that $xy \in P$, $x, y \in K$ implies that either $x^n \in P$ for some $n \geq 1$ or $y \in P$. It was shown in Badawi (2002, Theorem 3.4) that an integral domain R is an almost pseudo-valuation domain if and only if R is quasilocal with maximal ideal M such that for every nonzero $x \in K$, either $x^n \in M$ for some positive integer $n \geq 1$ or $ax^{-1} \in M$ for every nonunit $a \in R$. Hence it is clear that a pseudo-valuation domain is an almost pseudo-valuation domain, however Badawi (2002, Example 3.9) is an example of an almost pseudo-valuation domain which is not a pseudo-valuation domain. Anderson and Zafrullah (1991) introduced the notion of an *almost valuation domain*. Recall from Anderson and Zafrullah (1991) that an integral domain R is said to be an *almost valuation domain* if for every nonzero $x \in K$, there exists an $n \geq 1$ (depending on x) with x^n or $x^{-n} \in R$.

In this article, we introduce a new closely related class of integral domains. We define a prime ideal P of R to be a *pseudo-strongly prime ideal* if, whenever $x, y \in K$ and $xyP \subseteq P$, then there is a positive integer $m \geq 1$ such that either $x^m \in R$ or $y^m P \subseteq P$. If each prime ideal of R is a pseudo-strongly prime ideal, then R is called a *pseudo-almost valuation domain* (*PAVD*). We show in Theorem 2.8 that an integral domain R is a *PAVD* if and only if for every nonzero $x \in K$, there is a positive integer $n \geq 1$ such that either $x^n \in R$ or $ax^{-n} \in R$ for every nonunit $a \in R$. Thus it is clear that an almost valuation domain is a *PAVD* and an almost pseudo-valuation domain is a PAVD; however we show in Example 3.6 that for each $n \geq 1$, there is a *PAVD* with Krull dimension n which is neither an almost valuation domain nor an almost pseudo-valuation domain. We show in Proposition 2.2 that a *PAVD* is quasilocal with linearly ordered prime ideals and we give an example (Example 3.4) of a quasilocal domain R with linearly ordered prime ideals but R is not a *PAVD*. We then have the following implications, none of which is reversible:

> valuation domain $\Rightarrow$ pseudo-valuation domain $\Rightarrow$ almost pseudo-valuation domain $\Rightarrow$ pseudo-almost valuation domain $\Rightarrow$ quasilocal domain with linearly ordered prime ideals **AND** valuation domain $\Rightarrow$ almost valuation domain $\Rightarrow$ pseudo-almost valuation domain $\Rightarrow$ quasilocal domain with linearly ordered prime ideals.

Among the properties of *PAVD*s which will be studied in this article, we show in Proposition 2.16 that if an integral domain R admits a nonzero principal pseudo-strongly prime ideal P, then R is an almost valuation domain with maximal ideal P; we show in Theorem 2.15 that a quasilocal domain R with maximal ideal M is a *PAVD* if and only if $V = (M : M)$ is an almost valuation domain with maximal ideal $Rad(MV)$ (the radical of MV in V); we show (Theorem 4.10) that the integral closure of a *PAVD* with maximal ideal M is a valuation domain if and only if $(M : M)$ is integral over R; we show in Propositions 2.22 and 4.11 that a Noetherian *PAVD* has Krull dimension ≤ 1 and R' is a valuation domain; we show in Theorem 2.19 that a *PAVD* is a pullback of an almost valuation domain; we show in Corollary 4.12 that every overring of a *PAVD* R is a *PAVD* if and only if R' is a valuation domain and every integral overring of R is a *PAVD*. Let V be a valuation domain of the form $F + M$, where F is a field and M is the maximal ideal of V. Let D be a proper subring of F, and set $R = D + M$. Then, we show in Theorem 3.1 that R is a *PAVD* if and only if either D is a field or D is a *PAVD* with quotient field H such that $H \subseteq F$ is a root extension. (Recall that an extension $R \subseteq B$ with the property that for each $b \in B$ there exists a $n \geq 1$ (depending on b) such that $b^n \in R$ is called a *root extension*.)

2. PROPERTIES OF PAVDs

We start this section with the following definition.

Definition. A prime ideal P of R is called a *pseudo-strongly prime ideal* if, whenever $x, y \in K$ and $xyP \subseteq P$, there is a positive integer $m \geq 1$ such that either $x^m \in R$ or $y^m P \subseteq P$. If every prime ideal P of R is a pseudo-strongly prime ideal, then R is a *PAVD*.

Let S be a subset of an integral domain R with quotient field K. Then $E(S) = \{x \in K \mid x^n \notin S \text{ for every } n \geq 1\}$. We have the following lemma which is an analog of Hedstrom and Houston (1978a, Proposition 1.2).

Lemma 2.1. *Let P be a prime ideal of R. Then P is a pseudo-strongly prime ideal if and only if for every $x \in E(R)$, there is an $n \geq 1$ such that $x^{-n}P \subseteq P$.*

Proof. Suppose that P is a pseudo-strongly prime ideal. Let $x \in E(R)$. Then $xx^{-1}P = P$. Since $x \in E(R)$, there is an $n \geq 1$ such that $x^{-n}P \subseteq P$. Conversely, suppose that for every $x \in E(R)$ there is an $n \geq 1$ such that $x^{-n}P \subseteq P$. Let $x, y \in K$ such that $xyP \subseteq P$. Suppose that $x \in E(R)$. Then by hypothesis, there is an $n \geq 1$ such that $x^{-n}P \subseteq P$. Since $x^n y^n P \subseteq P$ and $x^{-n}P \subseteq P$, we conclude that $y^n P = x^{-n}(x^n y^n P) \subseteq x^{-n}P \subseteq P$. □

Proposition 2.2. *Let R be a PAVD. Then the prime ideals of R are linearly ordered. In particular, R is quasilocal.*

Proof. Suppose that P and Q are two distinct prime ideals of R such that neither $P \subseteq Q$ nor $Q \subseteq P$. Then there is a $p \in P \backslash Q$ and there is a $q \in Q \backslash P$. Set $x = p/q$. Since $p \notin Q$, we conclude that $x \in E(R)$. Hence, by Lemma 2.1, there is an $n \geq 1$ such that $ax^{-n} \in Q$ for every $a \in Q$. In particular, $q^{n+1}/p^n = qx^{-n} \in Q$. Thus, $q^{n+1} \in P$, and therefore $q \in P$, which is a contradiction. Hence, either $P \subseteq Q$ or $Q \subseteq P$. □

Example 3.4 shows that the converse of the above proposition is not true.

Proposition 2.3. *Let P be a pseudo-strongly prime ideal of R. Then for every $p \in P$ and for every $r \in R \backslash P$ there is an n (depending on p and r) such that $p^n/r^n \in P$.*

Proof. Let $p \in P$ and $r \in R \backslash P$. Set $x = r^2/p$. Since $r \notin P$, we conclude that $x \in E(R)$. Hence, by Lemma 2.1, there is an $n \geq 1$ such that $ax^{-n} \in P$ for every $a \in P$. In particular, $p^{2n}/r^{2n} = p^n x^{-n} \in P$. □

Proposition 2.4. *Let P be a pseudo-strongly prime ideal of R. Suppose that P contains a prime ideal Q of R. Then for every $q \in Q$ and for every $p \in P \backslash Q$, there is an $n \geq 1$ (depending on q and p) such that $q^n/p^n \in Q$.*

Proof. Let $q \in Q$ and $p \in P \backslash Q$. Set $x = q/p$. Suppose that $x^n \notin Q$ for every $n \geq 1$. Then, $x \in E(R)$. Thus, by Lemma 2.1, there is an $n \geq 1$ such that $x^{-n}P \subseteq P$. In particular, $p^{n+1}/q^n = px^{-n} \in P$. Thus, $p \in Q$, which is a contradiction. Hence, there is an $n \geq 1$ such that $q^n/p^n = x^n \in Q$. □

It is known (Hedstrom and Houston, 1978a, Theorem 1.4) that an integral domain R is a pseudo-valuation domain if and only if some maximal ideal of R is a strongly prime ideal; also, it is shown (Badawi, 2002, Theorem 3.4) that an integral domain R is an almost pseudo-valuation domain if and only if a maximal ideal of R is a strongly primary ideal. We have a similar result for *PAVD*s.

Theorem 2.5. *An integral domain R is a PAVD if and only if some maximal ideal of R is a pseudo-strongly prime ideal.*

Proof. Suppose that a maximal ideal M of R is a pseudo-strongly prime ideal. First, we show that R is quasilocal. Hence, suppose that N is a maximal ideal of R such that $M \neq N$. Since, by Proposition 2.3, for every $m \in M$ and $b \in N \backslash M$ there is an $n \geq 1$ such that $m^n/b^n \in M$, we conclude that $M \subseteq N$, which is impossible. Thus, R is a quasilocal domain with maximal ideal M. Now, let P be a prime ideal of R. By Lemma 2.1, we need only show that for every $x \in E(R)$, there is an $n \geq 1$ such that $px^{-n} \in P$ for every $p \in P$. Hence, let $x \in E(R)$. Since M is a pseudo-strongly prime ideal, there is an $n \geq 1$ such that $ax^{-n} \in M$ for every $a \in M$. In particular, $px^{-n} \in M$ for every $p \in P$. Suppose that $px^{-n} = b \in M \backslash P$ for some $p \in P$. Then by Proposition 2.4, there is an $m \geq 1$ such that $p^m/b^m \in P$. Thus, $px^{-n} = b$ implies that $(p^m/b^m)x^{-nm} = 1$. Thus, $x^{nm} = p^m/b^m \in P$. Hence, $x \notin E(R)$, which is a contradiction. Thus, $px^{-n} \in P$ for every $p \in P$. □

In light of the proof of Theorem 2.5, we have the following corollary.

Corollary 2.6. *Let P be a pseudo-strongly prime ideal of R. If Q is a prime ideal of R and $Q \subseteq P$, then Q is a pseudo-strongly prime ideal of R.*

For the proof of our next result, we need the following proposition.

Proposition 2.7 (Badawi, 1995, Theorem 1(5)). *Let R be an integral domain. Then the prime ideals of R are linearly ordered (and hence R is quasilocal) if and only if for every nonzero nonunit elements a, b of R, either $a \mid b^n$ or $b \mid a^n$ for some $n \geq 1$.*

Theorem 2.8. *An integral domain R is a PAVD if and only if for every $x \in E(R)$, there is an $n \geq 1$ such that $ax^{-n} \in R$ for every nonunit $a \in R$.*

Proof. Suppose that R is a *PAVD*. Then R is quasilocal by Proposition 2.2. Let M be the maximal ideal of R, and suppose that $x \in E(R)$. Then, by Lemma 2.1, there is an $n \geq 1$ such that $x^{-n}M \subseteq M \subseteq R$. Conversely, suppose that for every $x \in E(R)$, there is an $n \geq 1$ such that $ax^{-n} \in R$ for every nonunit a of R. First, we show that R is quasilocal. Let a, b be nonzero nonunit elements of R. Suppose that $a \nmid b^n$ in R for every $n \geq 1$. Then $x = b/a \in E(R)$. Hence, by hypothesis, there is an $n \geq 1$ such that $cx^{-n} \in R$ for every nonunit c of R. In particular, $a^{n+1}/b^n = ax^{-n} \in R$. Thus, $b \mid a^{n+1}$ in R. Thus, by Proposition 2.7, the prime ideals of R are linearly ordered. Hence, R is quasilocal. Let M be the maximal ideal of R. Suppose that $x \in E(R)$. Thus, by hypothesis, there is an $n \geq 1$ such that $ax^{-n} \in R$ for every $a \in M$. Since M is the maximal ideal of R and $x \in E(R)$, we conclude that $ax^{-n} \in M$ for every $a \in M$. By Lemma 2.1, M is a pseudo-strongly prime ideal of R. Hence, R is a *PAVD* by Theorem 2.5. □

The following proposition is a restatement of the above theorem.

Proposition 2.9. *An integral domain R is a PAVD if and only if for every $a, b \in R$ either $a^n \mid b^n$ in R for some $n \geq 1$ or there is an $m \geq 1$ such that $b^m \mid ca^m$ in R for every nonunit c of R.*

We recall the following proposition.

Proposition 2.10.

(i) (*Hedstrom and Houston,* 1978a, *Theorem* 1.5(3)). *A quasilocal domain (R, M) is a pseudo-valuation domain if and only if $x^{-1}M \subseteq M$ for every $x \in K \backslash R$.*
(ii) (*Badawi,* 2002, *Lemma* 2.3 *and Theorem* 3.4). *An integral domain R is an almost pseudo-valuation domain if and only if R is quasilocal with maximal ideal M such that $x^{-1}M \subseteq M$ for every $x \in E(M)$.*

In view of the above proposition, it is clear that a pseudo-valuation domain is an almost pseudo-valuation domain. The proof of the following result is clear by Proposition 2.10 and Theorem 2.8.

Proposition 2.11. *Suppose that R is an almost pseudo-valuation domain. Then R is a PAVD. In particular, if R is a pseudo-valuation domain, then R is a PAVD.*

The proof of the following result is also clear by the definition of almost valuation domain and Theorem 2.8.

Proposition 2.12. *Suppose that R is an almost valuation domain. Then R is a PAVD.*

We show in Example 3.6 that for each $n \geq 1$, there is a *PAVD* with Krull dimension n that is neither an almost valuation domain nor an almost pseudo-valuation domain. Hence we have the following implications, none of which is reversible:

> valuation domain $\Rightarrow$ pseudo-valuation domain $\Rightarrow$ almost pseudo-valuation domain $\Rightarrow$ pseudo-almost valuation domain $\Rightarrow$ quasilocal domain with linearly ordered prime ideals **AND** valuation domain $\Rightarrow$ almost valuation domain $\Rightarrow$ pseudo-almost valuation domain $\Rightarrow$ quasilocal domain with linearly ordered prime ideals.

In the following result, we show that a root closed *PAVD* is a pseudo-valuation domain. Recall that R is called *root closed* if, whenever $x \in K$ and $x^n \in R$ for some $n \geq 1$, then $x \in R$.

Theorem 2.13. *Let R be a root closed PAVD. Then R is a pseudo-valuation domain.*

Proof. Let M be the maximal ideal of R. Suppose that $x \in K \backslash R$. Since R is root closed, $x \in E(R)$. Hence, there is an $n \geq 1$ such that $x^{-n}M \subseteq M$ by Lemma 2.1. Thus, $x^{-n}m^n \in R$ for every $m \in M$. Hence, once again, since R is root closed, we conclude that $x^{-1}m \in R$ for every $m \in M$. Thus, R is a pseudo-valuation domain by Proposition 2.10(1). □

It is known (Dobbs, 1978, Lemma 4.5(i)) that if P is a prime ideal of a pseudo-valuation domain R, then R/P is a pseudo-valuation domain. We have a similar result for *PAVD*s.

Proposition 2.14. *Let R be a PAVD and P be a prime ideal of R. Then $D = R/P$ is a PAVD.*

Proof. Let R be a PAVD and P be a prime ideal of R. Set $D = R/P$ and let $x, y \in D$. Then $x = a + P$ and $y = b + P$ for some $a, b \in R$. Suppose that $x^n \nmid y^n$ in D for every positive integer $n \geq 1$. Then, $a^n \nmid b^n$ in R for every positive integer $n \geq 1$. Thus, by Proposition 2.9, there is a positive integer $m \geq 1$ such that $b^m \mid ca^m$ in R for every nonunit c of R. Thus, $y^m \mid zx^m$ for every nonunit z of D. Hence, by Proposition 2.9, D is a *PAVD*. □

It is known (Anderson and Dobbs, 1980, Proposition 2.5) that a quasilocal domain R with maximal ideal M is a pseudo-valuation domain if and only if $(M : M)$ is a valuation domain with maximal ideal M. We have the following result.

Theorem 2.15. *A quasilocal domain R with maximal ideal M is a PAVD if and only if $V = (M : M)$ is an almost valuation domain with maximal ideal $Rad(MV)$.*

Proof. Suppose that R is a *PAVD*. Let $x \in E(V)$. Then $x \in E(R)$. Hence, there is an $n \geq 1$ such that $x^{-n}M \subseteq M$. Thus, $x^{-n} \in V$. Thus, V is an almost valuation domain. Now, let x be a nonunit of V. Suppose that $x \notin Rad(MV)$. Then, since $MV = M$ and x is a nonunit of V, we conclude that $x \in E(R)$. Thus, by Lemma 2.1, there is an $n \geq 1$ such that $x^{-n}M \subseteq M$. Hence, $x^{-n} \in V$. Since $x \in V$ and $x^{-n} \in V$, we conclude that x is a unit of V, which is a contradiction. Thus, if x is a nonunit of V, then $x \in Rad(MV)$. Hence, $Rad(MV)$ is the maximal ideal of V. Conversely, suppose that $V = (M : M)$ is an almost valuation domain with maximal ideal $Rad(MV)$. Suppose that $x \in E(R)$. Then, $x \notin Rad(MV)$. Thus, if $x^n \in V$ for some $n \geq 1$, then x^n is a unit of V, and hence $x^{-n}M \subseteq M$. Thus, suppose that $x \in E(V)$. Hence, since V is an almost valuation domain, $x^{-m} \in V$ for some $m \geq 1$. Thus, $x^{-m}M \subseteq M$. Hence, M is a pseudo-strongly prime ideal by Lemma 2.1. Thus, R is a $PAVD$ by Theorem 2.5. □

Before we state our next result, we would like to point out that if R is a $PAVD$ with maximal ideal M, then $(M : M)$ need not be a valuation domain, see Example 3.5. It was shown Anderson (1983, Proposition 4.3(2)) that if an integral domain R admits a nonzero principal strongly prime ideal, then R is a valuation domain. We have the following result.

Proposition 2.16. *Suppose that an integral domain R admits a nonzero principal pseudo-strongly prime ideal P. Then R is an almost valuation domain with maximal ideal P.*

Proof. Suppose that P is a nonzero principal pseudo-strongly prime ideal of R. Then $P = (p)$ for some nonzero prime element $p \in R$. Now, suppose that P is a nonmaximal ideal of R. Then there is a nonunit $r \in R \backslash P$. Hence by Proposition 2.3, let n be the least positive integer such that $p^n = r^n d$ for some nonunit $d \in R$. Since $r \notin P$, $p \mid d$ in R. Suppose that $n = 1$. Then r is a unit of R which is a contradiction. Hence suppose that $n > 1$. Then $p^{n-1} = r^{n-1}(rd/p)$, which is again a contradiction since n is the least positive integer such that $r^n \mid p^n$. Thus P is a maximal ideal of R. Hence R is a $PAVD$ by Theorem 2.5 and thus $(P : P)$ is an almost valuation domain by Theorem 2.15. Since P is a nonzero principal ideal, we have $(P : P) = R$ is an almost valuation domain. □

Recall that an overring V of R is said to be a root extension of R if for every $x \in V$, there is an $n \geq 1$ such that $x^n \in R$.

Theorem 2.17. *Let R be a quasilocal domain with maximal ideal M. Suppose that V is an almost valuation overring of R such that M is an ideal of V and $Rad(M)$ (in V) is the maximal ideal of V. Then R is an almost valuation domain if and only if V is a root extension of R.*

Proof. If $V = R$, then there is nothing to prove. Hence, we assume that $V \neq R$. Suppose that R is an almost valuation domain. Let $x \in V \backslash R$. If $x \in Rad(M)$, then $x^k \in M \subset R$ for some $k > 1$. Hence, assume that $x \notin Rad(M)$. Since $Rad(M)$ is the maximal ideal of V, we conclude that x is a unit of V. Since R is an almost valuation domain and x is a unit of $V \backslash R$, we conclude that x^n is a unit of R for some $n > 1$. Thus, V is a root extension of R. Conversely, suppose that V is a root extension of R and $x \in E(R)$. Since V is a root extension of R, we conclude that $x \in E(V)$. Since V is an almost valuation domain and $x \in E(V)$, there is an $n \geq 1$ such that $x^{-n} \in V$. Since V is a root extension of R and $x^{-n} \in V$, we have $x^{-nm} \in R$ for some $m \geq 1$. Hence, R is an almost valuation domain. □

We recall the following result.

Proposition 2.18 (Anderson and Dobbs, 1980, Lemma 3.1). *Let R and V be integral domains such that $R \subseteq V$. If R and V share a nonzero ideal, then R and V have the same quotient field.*

It is known (Anderson and Dobbs, 1980, Proposition 2.6) that a pseudo-valuation domain is a pullback of a valuation domain. An excellent article on pullbacks is Fontana (1980). Our next result shows how to construct a *PAVD* as a pullback of an almost valuation domain.

Theorem 2.19. *Let V be an almost valuation domain with nonzero maximal ideal N and let M be an ideal of V such that $Rad(M) = N$, $F = V/M$, $\alpha : V \longrightarrow F$ the canonical epimorphism, H be a field contained in F, and $R = \alpha^{-1}(H)$. Then the pullback $R = V \times_F H$ is a PAVD with maximal ideal M. In particular, if H is properly contained in F and V is not a root extension of R, then R is a PAVD which is not an almost valuation domain.*

Proof. In view of the construction stated in the hypothesis, it is well known that for any domain V, M is a maximal ideal of R. Also, it is clear that R and V have the same quotient field by Proposition 2.18. Suppose that $x \in E(R)$. Then, since V is an almost valuation domain, either $x^n \in V\backslash R$ for some $n \geq 1$ or $x \in E(V)$. Hence, suppose that $x^n \in V\backslash R$ for some $n \geq 1$. Since M is an ideal of V, $Rad(M) = N$ is the maximal ideal of V and $x \in E(R)$, we conclude that x^n is a unit of V. Thus, $x^{-n}M \subseteq M$. Now, suppose that $x \in E(V)$. Then, since V is an almost valuation domain, we conclude that $x^{-n} \in V$ for some $n \geq 1$. Since M is an ideal of V, we conclude that $x^{-n}M \subseteq M$. Thus, M is a pseudo-strongly prime ideal of R by Lemma 2.1. Hence, R is a *PAVD* by Theorem 2.5. The remaining part is clear from Theorem 2.17. □

In view of Theorems 2.19 and 2.17, the following is an example of a *PAVD* which is not an almost valuation domain.

Example 2.20. Let F be a finite field, $H = F(X)$ be the quotient field of $F[X]$, $V = H + Y^3H[[Y]]$, and $R = F + Y^3H[[Y]]$. Then $V' = H + YH[[Y]]$ is a valuation domain and $V \subseteq V'$ is a root extension. Thus V is an almost valuation domain by Anderson and Zafrullah (1991, Theorem 5.6). It is clear that V is not a valuation domain. Since $M = Y^3H[[Y]]$ is the maximal ideal of both domains V and R, we have $F = R/M$ is a subfield of $V/M = H$. Let $\alpha : V \longrightarrow H = V/M$. Then $\alpha^{-1}(F = R/M) = R$ is a *PAVD* by Theorem 2.19. However, since V is not integral over R, R is not an almost valuation domain again by Theorem 2.17.

In light of Theorems 2.15 and 2.19, we have the following corollary which is an analog of Anderson and Dobbs (1980, Proposition 2.6).

Corollary 2.21. *The pseudo-almost valuation domains are precisely the pullbacks in the category of commutative rings (with 1) of diagrams of the form:*

$$\begin{array}{c} V \\ \downarrow \\ F \leftarrow H, \end{array}$$

in which V is an almost valuation domain having maximal ideal $Rad(M)$ for some ideal M of V, $F = V/M$, the vertical map is the canonical surjection, H is a field contained in F, and the horizontal map is inclusion.

In the following result, we show that a Noetherian *PAVD* has Krull dimension ≤ 1. In Example 3.6, we show that for every $n \geq 1$ (possibly infinite), there is a *PAVD* with Krull dimension n.

Proposition 2.22. *If R is a Noetherian PAVD, then R has Krull dimension ≤ 1.*

Proof. (It is similar to the proof of Hedstrom and Houston, 1978a, Proposition 3.2.) This follows from the fact that if $P_1 \subseteq P_2 \subseteq P_3$ are three distinct prime ideals of a Noetherian domain, then there are infinitely many prime ideals between P_1 and P_3 (Kaplansky, 1974, Theorem 144), and the fact that the prime ideals of R are linearly ordered (Proposition 2.2). □

3. EXAMPLES AND $D + M$ CONSTRUCTION OF *PAVDs*

Recall that an extension $R \subseteq B$ with the property that for each $b \in B$ there exists a $n \geq 1$ (depending on b) such that $b^n \in R$ is called a *root extension*.

Theorem 3.1. *Let V be a valuation domain of the form $F + M$, where F is a field and M is the maximal ideal of V. Let D be a proper subring of F. Set $R = D + M$. Then R is a PAVD if and only if either D is a field or D is a PAVD with quotient field H such that $H \subseteq F$ is a root extension.*

Proof. Suppose that R is a *PAVD* and assume that D is not a field. By Proposition 2.14, $R/M \cong D$ is a *PAVD*. Let N be the maximal ideal of D, and H be the quotient field of D. Since $D \subseteq F$ and F is a field, $H \subseteq F$. Hence, assume that H is properly contained in F. Let $x \in F \backslash H$. Then x is in the quotient field of R. Suppose that $x^n \notin H$ for every $n \geq 1$. Then $x \in E(R)$. Since x is in the quotient field of R, R is a *PAVD*, and $x \in E(R)$, we conclude that $x^{-n}(N + M) \subseteq (N + M)$ for some $n \geq 1$ by Lemma 2.1. Since F is a field, $x^{-n} \in F$, $N \subseteq H \subseteq F$, $M \cap F = \{0\}$, and $x^{-n}(N + M) \subseteq (N + M)$, we conclude that $x^{-n}N \subseteq N$. Thus, let d be a nonzero element of N. Since $x^{-n}d = g \in N$, we conclude that $x^n = d/g \in H$, which is a contradiction. Hence, $H \subseteq F$ is a root extension.

Conversely, suppose that D is a field. Then M is the maximal ideal of R. Since $(M : M) = V$ is a valuation domain, R is a PAVD (PVD) by Theorem 2.15. Thus, suppose that D is not a field and D is a *PAVD* with quotient field H such that $H \subseteq F$ is a root extension. Let N be the maximal ideal of D. Then, $N + M$ is the maximal ideal of R by Bastida and Gilmer (1973). Now, suppose that $x \in E(R)$. Since V is a valuation overring of R and M is the maximal ideal of V, we conclude that either $x^{-1} \in M$ or x is a unit of V. If $x^{-1} \in M$, then it is clear that $x^{-1}(N + M) \subseteq (N + M)$. Hence, suppose that x is a unit of V. Then, $x = f + d$ is a unit of V, where f is a nonzero element of F and $d \in M$. Since $H \subseteq F$ is a root extension and $x \in E(R)$, we conclude that $f^m \in H$ for some $m \geq 1$ and $f^{mn} \notin D$ for every $n \geq 1$. Thus, $f^{-mn}N \subseteq N$ for some $n \geq 1$ by Lemma 2.1. Hence, $x^{-nm}(N + M) \subseteq (N + M)$. Thus, $N + M$

is a pseudo-strongly prime ideal of R by Lemma 2.1. Thus, R is a $PAVD$ by Theorem 2.5. □

By an argument similar to the one just given in the proof of Theorem 3.1, one can prove the following result.

Corollary 3.2. *Let F, M, V, D, and R be as in Theorem* 3.1. *Then R is an almost valuation domain if and only if D is an almost valuation domain with quotient field H such that $H \subseteq F$ is a root extension.*

Let F, M, V, and D as in Theorem 3.1. Dobbs (1978, Proposition 4.9) showed that $R = D + M$ is a pseudo-valuation domain if and only if either D is a field or D is a pseudo-valuation domain with quotient field F. In view of the proof of Theorem 3.1, we will give an alternative proof of the part: If R is a pseudo-valuation domain and D is not a field, then F is the quotient field of D.

Proposition 3.3 (Dobbs, 1978, Proposition 4.9)**.** *Let F, M, V, and D be as in Theorem* 3.1. *If $R = D + M$ is a pseudo-valuation domain and D is not a field, then F is the quotient field of D.*

Proof. Deny. Let H be the quotient field of D, N be the maximal ideal of D, and K be the quotient field of R. Then, there is an $x \in F \backslash H$. Hence, $x \in K \backslash R$. Thus, $x^{-1}(N + M) \subseteq (N + M)$. Since F is a field, $x^{-1} \in F$, $N \subseteq H \subseteq F$, $F \cap M = \{0\}$, and $x^{-1}(N + M) \subseteq (N + M)$, we conclude that $x^{-1}N \subseteq N$. Thus, $x^{-1}a = b \in N$ for some nonzero $a \in N$. Hence, $x = a/b \in H$, which is a contradiction. Thus, $F = H$. □

Since a $PAVD$ is quasilocal with linearly ordered prime ideals by Proposition 2.2, the following is an example of a quasilocal Noetherian domain R with linearly ordered prime ideals that is not a $PAVD$.

Example 3.4. Let $R = C + CX^2 + x^4C[[X]] = C[[X^2, X^5]]$, where C is the field of complex numbers. Then R is a quasilocal Noetherian domain with maximal ideal $M = (X^2, X^5)R$, so its prime ideals are linearly ordered. Since $(M : M) = C[[X^2, X^3]]$ is not an almost valuation domain, we conclude that R is not a $PAVD$ by Theorem 2.15.

The following is an example of a $PAVD$ which is neither an almost valuation domain nor an almost pseudo-valuation domain.

Example 3.5. Let F be a finite field and $H = F(X)$ be the quotient field of $F[X]$. Set $R = F + HY^2 + Y^4H[[Y]]$. Then R is a quasilocal domain with maximal ideal $M = HY^2 + Y^4H[[Y]]$. Since $V = (M : M) = H + Y^2H[[Y]]$ is an almost valuation domain with maximal ideal $Rad(MV)$ in V, we conclude that R is a $PAVD$ by Theorem 2.15. It is clear that $V = (M : M) = H + Y^2H[[Y]]$ is not a valuation domain, and hence R is not an almost pseudo-valuation domain by Badawi (2002, Theorem 3.4). Also, since $F \subset H$ is not a root extension, R is not an almost valuation domain.

In the following example, we show that for every $n \geq 1$, there is a *PAVD* with Krull dimension n that is neither an almost valuation domain nor an almost pseudo-valuation domain.

Example 3.6. Let F be a finite field and $H = F(X)$ be the quotient field of $F[X]$. Set $D = F + HY^2 + Y^4H[[Y]]$. By Example 3.5, D is a *PAVD* with maximal ideal $M = HY^2 + Y^4H[[Y]]$ and Krull dimension one that is neither an almost valuation domain nor an almost pseudo-valuation domain. Let K be the quotient field of D and let $n > 1$. Then there is a valuation domain of the form $K + N$ with maximal ideal N and Krull dimension $n - 1$. Then $R = D + N$ is a *PAVD* by Theorem 3.1. By standard properties of the $D + M$-construction (see Bastida and Gilmer, 1973), R has Krull dimension n. Since D is not an almost valuation domain as it is shown in Example 3.5, R is not an almost valuation domain by Corollary 3.2. Since neither $Y \in (M + N : M + N)$ nor $1/Y \in (M + N : M + N)$, $(M + N : M + N)$ is not a valuation domain. Hence, R is not an almost pseudo-valuation domain by Badawi (2002, Theorem 3.4).

Recall from Theorem 2.15 that a quasilocal domain R with maximal ideal M is a *PAVD* if and only if $V = (M : M)$ is an almost valuation domain with maximal ideal $Rad(MV)$ in V. The following is an example of a *PAVD* R with maximal ideal M such that R is not an almost valuation domain and $(M : M)$ is an almost valuation domain that is not a valuation domain.

Example 3.7. Let F, H, R, and M be as in Example 3.5. Then R is a *PAVD* with maximal ideal M such that $(M : M)$ is an almost valuation domain that is not a valuation domain.

It is well known that if R is a pseudo-valuation domain with maximal ideal M, then R' (the integral closure of R in K) $\subseteq (M : M)$. The following is an example of a *PAVD* with maximal ideal M such that R' is a valuation domain, and $(M : M)$ is properly contained in R'.

Example 3.8. Let F be a finite field, $H = F + XF[[X]] = F[[X]]$, and $R = F + FX^2 + X^4F[[X]]$. Then R is a quasilocal domain with maximal ideal $M = FX^2 + X^4F[[X]]$. Since $V = (M : M) = F + X^2F[[X]]$ is an almost valuation domain with maximal ideal $Rad(MV)$ in V, we conclude that R is a *PAVD* by Theorem 2.15. Since $R' = H$ is a valuation domain, we conclude that $V = (M : M) = F + X^2F[[X]]$ is properly contained in R'.

The following is an example of a *PAVD* R with maximal ideal M such that neither $R' \subseteq (M : M)$ nor $(M : M) \subseteq R'$, and hence $(M : M)$ is not integral over R.

Example 3.9. Let F be a finite field and $H = F(X)$ be the quotient field of $F[X]$. Set $R = F + HY^2 + Y^4H[[Y]]$. Then R is a quasilocal domain with maximal ideal $M = HY^2 + Y^4H[[Y]]$. Since $V = (M : M) = H + Y^2H[[Y]]$ is an almost valuation domain with maximal ideal $Rad(MV)$ in V, we conclude that R is a *PAVD* by Theorem 2.15. Since H is not integral over F, $(M : M) = H + Y^2H[[Y]]$ is not integral over R, and hence $(M : M)$ is not contained in R'. Also, since Y is integral over R and $Y \notin (M : M)$, we conclude that R' is not contained in $(M : M)$.

4. OVERRINGS THAT ARE *PAVDs*

We start this section with the following result that is an analog of Anderson et al. (1995, Lemma 8).

Theorem 4.1. *Let R be a PAVD with maximal ideal M. Suppose that V is an overring of R such that $1/s \in V$ for some $s \in M$. Then V is an almost valuation domain.*

Proof. Suppose that $x \in E(V)$. Hence, $x \in E(R)$. Thus, by Lemma 2.1 there is an $n \geq 1$ such that $ax^{-n} \in M$ for every $a \in M$. In particular, $sx^{-n} = d \in M$. Hence, $x^{-n} = d/s$. Since $1/s \in V$, we conclude that $x^{-n} = d/s \in V$. Hence, V is an almost valuation domain. □

The following result is an analog of Hedstrom and Houston (1978a, Proposition 2.6).

Corollary 4.2. *Let R be a PAVD. Suppose that P is a nonmaximal prime ideal of R. Then R_P is an almost valuation domain.*

Proof. Since P is a nonmaximal prime ideal of R, we conclude that R_P contains an element of the form $1/s$ for some nonunit $s \in R\backslash P$. Hence, by Theorem 4.1, R_P is an almost valuation domain. □

The following result is an analog of Anderson (1979, Propositions 4.2 and 4.3)

Theorem 4.3. *Let P be a pseudo-strongly prime ideal of R. Then $(P : P)$ is an almost valuation domain. In particular, if R is a PAVD, then $(P : P)$ is an almost valuation domain for every prime ideal P of R.*

Proof. Let $B = (P : P)$. Suppose that $x \in E(B)$. Hence, $x \in E(R)$. Thus, by Lemma 2.1, there is an $n \geq 1$ such that $x^{-n}P \subseteq P$. Hence, $x^{-n} \in B$. Thus, B is an almost valuation domain. □

We recall the following result.

Proposition 4.4 (Anderson and Zafrullah, 1991, Theorem 5.6). *An integral domain R is an almost valuation domain if and only if R' is a valuation domain and $R \subseteq R'$ is a root extension.*

Proposition 4.5. *Let R be a PAVD with maximal ideal M. Set $V = (M : M)$. Then V' is a valuation domain (and hence is a pseudo-valuation domain) with maximal ideal $N = \{x \in K \mid x^n \in M \text{ for some } n \geq 1\}$, and $V \subseteq V'$ is a root extension. Furthermore, if B is an overring of R such that B does not contain an element of the form $1/s$ for some nonunit s of R, then $B \subseteq V'$. In particular, $R' \subseteq V'$.*

Proof. By Theorem 2.15, V is an almost valuation domain with maximal ideal $Rad(MV) = Rad(M)$ in V. Hence, by Proposition 4.4, V' is a valuation domain and $V \subseteq V'$ is a root extension. Thus, Let N be the maximal ideal of V'. Then, $N = \{x \in K : x^n \in Rad(MV)$ for some $n \geq 1\} = \{x \in K : x^m \in M$ for some $m \geq 1\}$. Now, suppose that B is an overring of R such that $1/s \notin B$ for every nonunit s of R. We will show that $B \subseteq V'$. Let $x \in B$. Suppose that $x \notin V'$. Since V is a valuation domain, $x^{-1} \in V$. Hence, x^{-1} is a nonunit of V'. Thus, $x^{-1} \in N$. Hence, $x^{-n} = d \in M$ for some $n \geq 1$. Hence, $x^n = 1/d \in B$ (since $x \in B$). A contradiction. Hence, $B \subseteq V'$. Since $1/m$ is never integral over R for every nonunit $m \in M$, we conclude that $R' \subseteq V'$. □

Recall from Hedstrom and Houston (1978a, Theorem 1.4) that a quasilocal integral domain with maximal ideal M is a pseudo-valuation domain if and only if M is a strongly prime ideal of R. It was shown in Badawi (2002, Proposition 3.7) that if R is an almost pseudo-valuation domain, then R' is a pseudo-valuation domain. For a *PAVD* we have a similar result.

Theorem 4.6. *Let R be a PAVD with maximal ideal M. Then R' is a pseudo-valuation domain with maximal ideal $N = \{x \in K \mid x^n \in M$ for some $n \geq 1\}$. Hence, R' and V' have the same maximal ideal, where V' is the integral closure of $V = (M : M)$.*

Proof. Let $V = (M : M)$. Since $R' \subseteq V'$ and N is a strongly prime ideal of V' by Proposition 4.5, we conclude that N is a strongly prime ideal of R'. Since $M \subseteq N$, by integrality, we conclude that N is the unique maximal ideal of R'. Thus, R' is a pseudo-valuation domain by Hedstrom and Houston (1978a, Theorem 1.4). □

Theorem 4.7. *Let R be a PAVD with maximal ideal M. If each overring of R is a PAVD, then $R' = V'$ is a valuation domain, where $V = (M : M)$.*

Proof. (It is similar to the proof of Badawi, 2002, Proposition 3.8.) Suppose that each overring of R is a *PAVD*. Since R' is a pseudo-valuation domain by Theorem 4.6, the proof of Hedstrom and Houston (1978a, Proposition 2.7) shows that if R' is not a valuation domain, then there is a nonquasilocal overring of R'. However, such an overring cannot be a *PAVD* by Proposition 2.2. Hence, R' is a valuation domain. Since $R' \subseteq V'$ by Proposition 4.5, and V' is a valuation domain with the same maximal ideal as R' by Theorem 4.6, we conclude that $R' = V'$. □

The converse of Theorem 4.7 is false, as the following example shows that.

Example 4.8. Let Q be the field of rational numbers and $F = Q(\sqrt{2})$. Set $V = F + XF[[X]] = F[[X]]$, $S = Q + QX + X^2F[[X]]$, and $R = Q + X^2F[[X]]$. Then R is a *PAVD* and $R' = V$ is a valuation domain, but S is an overring ring of R with maximal ideal $M = QX + X^2F[[X]]$ which is not a *PAVD* by Theorem 2.15. For let $a = 1 + \sqrt{2}$; then $a \in E(R)$ and $a^{-n}X \notin M$ for every $n \geq 1$. Thus, M is not a pseudo-strongly prime ideal of S by Lemma 2.1. Hence, S is not a *PAVD*.

In Example 4.8, S is an integral overring of R. The following result shows that this is the only stumbling block. A similar result for almost pseudo-valuation domains was obtained in Badawi (2002, Proposition 3.10).

Theorem 4.9. *Let R be a PAVD with R' a valuation domain, and assume that every integral overring of R is a PAVD. Then every overring of R is a PAVD.*

Proof. Let M be the maximal ideal of R, and set $V = (M : M)$. Since R' is a valuation domain, $R' \subseteq V'$ is a valuation domain by Proposition 4.5, and V' has the same maximal ideal as R' by Theorem 4.6, we have $R' = V'$. Now, let S be an overring of R. If S contains an element of the form $1/m$ for some $m \in M$, then R is an almost valuation domain by Theorem 4.1 and hence is a *PAVD* by Proposition 2.12. Thus, suppose that S does not contain an element of the form $1/m$ for some $m \in M$. Then $S \subseteq V'$ by Proposition 4.5. Hence, $S \subseteq R'$(since $R' = V'$). Thus, S is an integral overring of R. Hence, S is a *PAVD*. □

In view of Examples 3.8 and 3.9, we have the following result.

Theorem 4.10. *Let R be a PAVD with maximal ideal M. Then R' is a valuation domain if and only if $(M : M)$ is integral over R.*

Proof. Let $V = (M : M)$. Suppose that R' is a valuation domain. Then $R' = V'$ by the same argument as in the proof of Theorem 4.9. Hence, $(M : M)$ is integral over R. Conversely, suppose that V is integral over R. Hence, $R' = V'$ is a valuation domain by Proposition 4.5. □

In light of Theorem 4.10, we have the following result.

Proposition 4.11. *Let R be a PAVD with maximal ideal M. If M is finitely generated, then R' is a valuation domain. In particular, if R is a Noetherian PAVD, then R' is a valuation domain.*

Proof. Since M is finitely generated, $(M : M)$ is integral over R. Hence, R' is a valuation domain by Theorem 4.10. □

In view of Proposition 2.12, Theorem 4.1, Proposition 4.5, Theorem 4.7, Theorem 4.9, the proof of Theorems 4.9 and 4.10, we arrive at the following corollary.

Corollary 4.12. *Let R be a PAVD with maximal ideal M. Then the following conditions are equivalent:*

(1) *Every overring of R is a PAVD*;
(2) *$(M : M)$ is integral over R and every integral overring of R is a PAVD*;
(3) *R' is a valuation domain and every integral overring of R is a PAVD*;
(4) *If B is an overring of R that does not contain an element of the form $1/m$ for some nonunit m of R, then B is a PAVD.*

ACKNOWLEDGMENT

I would like to thank the referee for his great effort in proofreading the manuscript. In particular, I am grateful to the referee for his valuable comments on the examples of this article.

REFERENCES

Anderson, D. D., Zafrullah, M. (1991). Almost Bézout domains. *J. Algebra* 142:285–309.

Anderson, D. F. (1979). Comparability of ideals and valuation overrings. *Houston J. Math.* 5:451–463.

Anderson, D. F. (1983). When the dual of an integral domain is a ring. *Houston J. Math.* 9:325–332.

Anderson, D. F., Dobbs, D. E. (1980). Pairs of rings with the same prime ideals. *Can. J. Math.* 32:362–384.

Anderson, D. F., Badawi, A., Dobbs, D. E. (1995). *Pseudo-Valuation Rings*. Lecture Notes Pure Appl. Math. Vol. 171. New York/Basel: Marcel Dekker, pp. 155–161.

Badawi, A. (1995). On domains which have prime ideals that are linearly ordered. *Comm. Algebra* 23:4365–4373.

Badawi, A. (2002). Powerful ideals, strongly primary ideals, almost pseudo-valuation domains, and conducive domains. *Comm. Algebra* 30:1591–1606.

Bastida, E., Gilmer, R. (1973). Overrings and divisorial ideals of rings of the form $D + M$. *Michigan Math. J.* 20:79–95.

Dobbs, D. E. (1978). Coherence, ascent of going-down, and pseudo-valuation domains. *Houston J. Math.* 4:551–567.

Fontana, M. (1980). Topologically defined classes of commutative rings. *Ann. Mat. Pura Appl.* 123:331–355.

Hedstrom, J. R., Houston, E. G. (1978a). Pseudo-valuation domains. *Pacific J. Math.* 4:551–567.

Hedstrom, J. R., Houston, E. G. (1978b). Pseudo-valuation domains, II. *Houston J. Math.* 4:199–207.

Kaplansky, I. (1974). *Commutative Rings*. Chicago: The University of Chicago Press.