FACTORING NONNIL IDEALS INTO PRIME AND INVERTIBLE IDEALS

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Abstract. For a commutative ring R, let Nil(R) be the set of all nilpotent elements of R, $Z(R)$ be the set of all zero divisors of R, $T(R)$ be the total quotient ring of R, and $\mathcal{H} = \{R \mid R$ is a commutative ring and Nil(R) is a divided prime ideal of $R\}$. For a ring $R \in \mathcal{H}$, let $\phi : T(R) \longrightarrow R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and $b \in R \backslash Z(R)$. A ring R is called a *ZPUI ring* if every proper ideal of R can be written as a finite product of invertible and prime ideals of R. In this paper, we give a generalization of the concept of ZPUI domains which was extensively studied by Olberding to the context of rings that are in the class $\mathcal{H}$. Let $R \in \mathcal{H}$. If every nonnil ideal of R can be written as a finite product of invertible and prime ideals of R, then R is called a *nonnil-ZPUI ring*; if every nonnil ideal of $\phi(R)$ can be written as a finite product of invertible and prime ideals of $\phi(R)$, then R is called a *nonnil-ϕ-ZPUI ring*. We show that the theory of ϕ-ZPUI rings resembles that of ZPUI domains.

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, $Z(R)$ denotes the set of zero divisors of R, and $Nil(R)$ denotes the set of nilpotent elements of R. The elements in $R \backslash Z(R)$ are referred to as regular elements and an ideal I is said to be regular if it contains at least one regular element. For a nonzero ideal I, regular or not, we let $I^{-1} = \{x \in T(R) \mid xI \subset R\}$. An ideal I of a ring R is called *invertible* if $II^{-1} = R$.

We start by recalling some background materials. Recall from [12] and [5] that a prime ideal P of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \backslash P$; thus a divided prime ideal is comparable to every ideal of R. In [4], [6], [7], [9], and [8], the author investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring and Nil(R) is a divided prime ideal of $R\}$. Also, D.F.Anderson, Tom Lucas, and the author made further investigation on the class $\mathcal{H}$ in [2], [3], and most recently in [10]. In this paper, we give a generalization of the concept of factorization of ideals of an integral domain into a finite product of invertible and prime ideals which was extensively studied by Olberding [20] to the context of rings that are in the class $\mathcal{H}$. Observe that if R is an integral domain, then $R \in \mathcal{H}$. An ideal I of a ring R is said to be a *nonnil ideal* if $I \not\subseteq Nil(R)$. For each $R \in \mathcal{H}$, the map $\phi : T(R) \longrightarrow R_{Nil(R)}$ defined by $\phi(a/b) = a/b$ for each $a \in R$ and $b \in R \backslash Z(R)$ was introduced by the author in [4]. The map ϕ is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$ and ϕ restricted to R is a ring homomorphism from R into

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$R_{Nil(R)}$ given by $\phi(x) = x/1$ for each $x \in R$. Note that $T(\phi(R)) = R_{Nil(R)}$. Let $R \in \mathcal{H}$. Then R is said to be a *ϕ-ZPUI ring*, if each nonnil ideal I of $\phi(R)$ can be written as $I = JP_1P_2 \cdots P_n$, where J is an invertible ideal of $\phi(R)$ and $P_1, P_2, \ldots, P_n$ are prime ideals of $\phi(R)$. If every nonnil ideal I of R can be written as $I = JP_1P_2 \cdots P_n$, where J is an invertible ideal of R and $P_1, P_2, \ldots, P_n$ are prime ideals of R, then R is said to be a *nonnil-ZPUI ring*. Commutative ϕ-ZPUI rings that are in $\mathcal{H}$ are characterized in Theorem 2.9. Examples of ϕ-ZPUI rings that are not ZPUI rings are constructed in Theorem 2.13. It is shown in Theorem 2.14 that a ϕ-ZPUI ring is the pullback of a ZPUI domain. It is shown in Theorem 3.1 that a nonnil-ZPUI ring is a ϕ-ZPUI ring. Examples of ϕ-ZPUI rings that are not nonnil-ZPUI rings are constructed in Theorem 3.2.

If $Nil(R)$ is divided, then it is also the nilradical of $T(R)$ and the kernel of the map ϕ is also a common ideal of R and $T(R)$. Other useful features of each ring $R \in \mathcal{H}$ (see [4]) include the following: (i) $\phi(R) \in \mathcal{H}$, (ii) $T(\phi(R)) = R_{Nil(R)}$ is quasilocal with maximal ideal $Nil(\phi(R))$, (iii) $\phi(R)$ is naturally isomorphic to $R/Ker(\phi)$, (iv) $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$, and (v) $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R))$. If I is a nonnil ideal of a ring $R \in \mathcal{H}$, then observe that $Nil(R) \subset I$.

Throughout the paper we will use the technique of idealization of a module to construct examples. Recall that for an R-module B, the idealization of B over R is the ring formed from $R \times B$ by defining addition and multiplication as $(r, a)+(s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa)$, respectively. A standard notation for the "idealized ring" is $R(+)B$. See [17], [18] and [19] for basic properties of these rings.

2. ϕ-ZPUI RINGS

We recall the following two lemmas from [2].

Lemma 2.1. *(*[2, Lemma 2.3]*) Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$, and let I be an ideal of R. Then I is an invertible ideal of R if and only if $I/Nil(R)$ is an invertible ideal of $R/Nil(R)$.*

Lemma 2.2. *(*[2, Lemma 2.5]*) Let $R \in \mathcal{H}$ and let P be a prime ideal of R. Then R/P is ring-isomorphic to $\phi(R)/\phi(P)$. In particular, $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$*

Theorem 2.3. *Let $R \in \mathcal{H}$. Then R is a ϕ-ZPUI ring if and only if $R/Nil(R)$ is a ZPUI domain.*

Proof. Suppose that R is a ϕ-ZPUI ring. Set $D = \phi(R)/Nil(\phi(R))$, and let L be a nonzero ideal of D. Then $L = I/Nil(\phi(R))$ for some nonnil ideal I of $\phi(R)$. Thus $I = JP_1P_2 \cdots P_n$, where J is an invertible ideal of $\phi(R)$ and $P_1, P_2, \ldots, P_n$ are prime ideals of $\phi(R)$. Since $Nil(\phi(R)) = Z(\phi(R))$, we conclude that $J/Nil(\phi(R))$ is an invertible ideal of D by Lemma 2.1. Thus $L = I/Nil(\phi(R)) = (J/Nil(\phi(R)))(P_1/Nil(\phi(R))) \cdots (P_n/Nil(\phi(R)))$, and hence D is a ZPUI domain. Since D is ring-isomorphic to $R/Nil(R)$ by Lemma 2.2, we conclude that $R/Nil(R)$ is a ZPUI domain.

Conversely, suppose that $R/Nil(R)$ is a ZPUI domain. Then $D = \phi(R)/Nil(\phi(R))$ is a ZPUI domain by Lemma 2.2. Let I be a nonnil ideal of $\phi(R)$. Since $\phi(R) \in \mathcal{H}$, $I/Nil(\phi(R))$ is a nonzero ideal of D. Thus $I/Nil(\phi(R)) =$

$(J/Nil(\phi(R)))(P_1/Nil(\phi(R)))\cdots(P_n/Nil(\phi(R)))$, where J is an invertible ideal of $\phi(R)$ (by Lemma 2.1) and $P_1, P_2, \ldots, P_n$ are prime ideals of $\phi(R)$. We show that $I = JP_1P_2\cdots P_n$. This follows since $Nil(\phi(R)) \subset I$ because $Nil(\phi(R)) \subset P_i$ for each i and $Nil(\phi(R))$ is a divided prime ideal of $\phi(R)$. Thus R is a ϕ-ZPUI ring. □

Lemma 2.4. *Let $R \in \mathcal{H}$ and let I be a nonnil ideal of R. Then I is a finitely generated ideal of R if and only if $I/Nil(R)$ is a finitely generated ideal of $R/Nil(R)$.*

Proof. (*the proof is similar to the proof of* [9, Theorem 2.2].) Suppose that I is a nonnil finitely generated ideal of R. Since $Nil(R) \subset I$, it is clear that $I/Nil(R)$ is a finitely generated ideal of $R/Nil(R)$. Conversely, suppose that $J = I/Nil(R)$ is a finitely generated ideal of $R/Nil(R)$. Then $J = (i_1 + Nil(R), ..., i_n + Nil(R))$ for some i_m's in I. Since $Nil(R)$ is divided, we may assume that all the i_m's are nonnilpotent elements of R, and thus $Nil(R) \subset (i_1)$. Now let x be a nonnilpotent element of I. Then $x + Nil(R) = c_1i_1 + ... + c_ni_n + Nil(R)$ in $R/Nil(R)$ for some c_m's in R. Hence there is a $w \in Nil(R)$ such that $x + w = c_1i_1 + ... + c_ni_n$ in R. Since $x \in I \setminus Nil(R)$, $x \mid w$ in R. Thus $w = xf$ for some $f \in Nil(R)$. Hence $x + w = x + xf = x(1 + f) = c_1i_1 + ... + c_ni_n$ in R. Since $f \in Nil(R)$, $1 + f$ is a unit of R. Thus $x \in (i_1, ..., i_n)$, and hence I is a finitely generated ideal of R. □

Recall from [19] that a ring R is called a *Prüfer ring* if every finitely generated regular ideal of R is invertible. A Prüfer domain R is called a *strongly discrete Prüfer domain* as in [21] and [20] if R has no nonzero prime ideals P such that $P^2 = P$. A ring $R \in \mathcal{H}$ is said to be a *ϕ-Prüfer ring* as in [2] if $\phi(R)$ is a Prüfer ring. We call a ring $R \in \mathcal{H}$ a *nonnil-strongly discrete ring* if R has no nonnil prime ideal P such that $P^2 = P$. An integral domain R is called h-local as in [21] if each nonzero ideal of R is contained in at most finitely many maximal ideals of R and each nonzero prime ideal P of R is contained in a unique maximal ideal of R. A ring $R \in \mathcal{H}$ is said to be *nonnil-h-local* if each nonnil ideal of R is contained in at most finitely many maximal ideals of R and each nonnil prime ideal P of R is contained in a unique maximal ideal of R.

The reader can easily verify the following two lemmas.

Lemma 2.5. *Let $R \in \mathcal{H}$. Then R is a nonnil-h-local ring if and only if $R/Nil(R)$ is an h-local domain.*

Lemma 2.6. *Let $R \in \mathcal{H}$. Then R is a nonnil-strongly discrete Prüfer ring if and only if $R/Nil(R)$ is a strongly discrete Prüfer domain.*

We recall the following result from [2].

Proposition 2.7. *(*[2]*) Let $R \in \mathcal{H}$. Then R is a ϕ-Prüfer ring if and only if $R/Nil(R)$ is a Prüfer domain.*

Combining Lemmas 2.5 and 2.6 with Proposition 2.7, we arrive at the following result.

Proposition 2.8. *Let $R \in \mathcal{H}$. Then R is a nonnil-strongly discrete nonnil-h-local ϕ-Prüfer ring if and only if $R/Nil(R)$ is a strongly discrete h-local Prüfer domain.*

Since the class of integral domains is a subset of $\mathcal{H}$, the following result is a generalization of [20, Theorem 2.3].

Theorem 2.9. *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

1. *R is a ϕ-ZPUI ring;*
2. *Every nonnil proper ideal of R can be written as a product of prime ideals of R and a finitely generated ideal of R;*
3. *Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$;*
4. *R is a nonnil-strongly discrete nonnil-h-local ϕ-Prüfer ring.*

Proof. Set $D = R/Nil(R)$. **(1) ⇒ (2)**. Since R is a ϕ-ZPUI ring, D is a ZPUI domain by Theorem 2.3. Let I be a nonnil proper ideal of R. Then by [20, Theorem2.3] we have $I/Nil(R) = J/Nil(R)P_1/Nil(R)\cdots P_n/Nil(R)$, where J is a (nonnil) finitely generated ideal of R (by Lemma 2.4) and $P_1, P_2, \ldots, P_n$ are prime ideals of R. Since $Nil(R)$ is divided, it is easily verified that $I = JP_1\cdots P_n$.

(2) ⇒ (3). Let L be a nonnil proper ideal of $\phi(R)$. Then $L = \phi(I)$ for some nonnil proper ideal I of R. Since $I = JP_1\cdots P_n$, where J is a (nonnil) finitely generated ideal of R and $P_1, P_2, \ldots, P_n$ are prime ideals of R, it is easily verified that $L = \phi(I) = \phi(J)\phi(P_1)\cdots\phi(P_n)$, where $\phi(J)$ is a finitely generated ideal of $\phi(R)$ and $\phi(P_1), \ldots, \phi(P_n)$ are (nonnil) prime ideals of $\phi(R)$.

(3) ⇒ (4). Let $F = \phi(R)/Nil(\phi(R))$. Then every nonzero ideal of F can be written as a product of prime ideals of F and a finitely generated ideal of F, and thus F is a strongly discrete h-local Prüfer domain by [20, Theorem2.3]. Since F is ring-isomorphic to D, we conclude that D is a strongly discrete h-local Prüfer domain, and hence R is a nonnil-strongly discrete nonnil-h-local ϕ-Prüfer ring by Proposition 2.8.

(4) ⇒ (1). Since R is a nonnil-strongly discrete nonnil-h-local ϕ-Prüfer ring, we conclude that $D = R/Nil(R)$ is a strongly discrete h-local Prüfer domain by Proposition 2.8. Thus D is a ZPUI domain by [20, Theorem2.3], and hence R is a ϕ-ZPUI ring by Theorem 2.3. □

Let $R \in \mathcal{H}$ such that $Z(R) = Nil(R)$. Then $\phi(R) = R$, and hence R is a ϕ-ZPUI ring if and only if R is a nonnil-ZPUI ring. We state this connection in the following corollary.

Corollary 2.10. *Let $R \in \mathcal{H}$ such that $Nil(R) = Z(R)$. The following statements are equivalent:*

1. *R is a nonnil-ZPUI ring;*
2. *R is a ϕ-ZPUI ring;*
3. *Every nonnil proper ideal of R can be written as a product of prime ideals of R and a finitely generated ideal of R;*
4. *R is a nonnil-strongly discrete nonnil-h-local Prüfer ring.*

Recall that a *special primary ring* is a quasilocal commutative ring R with maximal ideal M such that every proper ideal of R is a power of M. We state the following useful lemma.

Lemma 2.11. *(see [20, Lemma 3.2 and Theorem 3.3]). Let $R \in \mathcal{H}$. Then R is a ZPUI ring if and only if R is either a strongly discrete h-local Prüfer domain or a special primary ring.*

Proof. . Suppose that R is a ZPUI ring. First observe that if a ring $A \cong A_1 \oplus \cdots \oplus A_n$ (where each A_i is a ring with $1 \neq 0$) and $n \geq 2$, then $Nil(A)$ is never divided, and hence $A \notin \mathcal{H}$. Now since R is a ZPUI ring, by [20, Theorem 3.3] we have $R \cong D_1 \oplus \cdots \oplus D_n$, where each D_i is either a strongly discrete h-local Prüfer domain or a special primary ring. Since $Nil(R)$ is divided, by the observation we just stated, we conclude that $n = 1$, and thus R is either a strongly discrete h-local Prüfer domain or a special primary ring. The converse is clear by [20, Theorem 3.3]. □

Our non-domain examples of ϕ-ZPUI rings that are not ZPUI rings are provided by the idealization construction $R(+)B$ arising from a ring R and an R-module B as in [19, Chapter VI]. We recall this construction. Let $R(+)B = R \times B$, and define:

(1) $(r, b) + (s, c) = (r + s, b + c)$.
(2) $(r, b)(s, c) = (rs, sb + rc)$.

Under these definitions $R(+)B$ becomes a commutative ring with identity. We recall the following two facts.

Proposition 2.12. *Let R be a ring, B be an R-module, and $Z(B)$ be the set of zerodivisors on B. Then:*

(1) *(*[19, Theorem 25.1]*)The ideal J of $R(+)B$ is prime (respectively, maximal) if and only if $J = P(+)B$, where P is a prime (respectively, maximal) ideal of R, and hence the Krull dimension of R is equal to the Krull dimension of $R(+)B$;*
(2) *(*[19, Theorem 25.3]*) $(r, b) \in Z(R(+)B)$ if and only if $r \in Z(R) \cup Z(B)$.*

Olberding in [20, Corollary 2.4] showed that for each $n \geq 1$, there exists a ZPUI domain with Krull dimension n. A Dedekind domain is a trivial example of a ZPUI domain.

Theorem 2.13. *Let A be a ZPUI domain (i.e. A is a strongly discrete h-local Prüfer domain by* [20, Theorem2.3]*) with Krull dimension $n \geq 1$ and quotient field F, and let K be an extension ring of F (i.e. K is a ring and $F \subseteq K$). Then $R = A(+)K \in \mathcal{H}$ is a ϕ-ZPUI ring with Krull dimension n that is not a ZPUI ring.*

Proof. It is easy to see that $Nil(R) = \{0\}(+)K$. We show that $Nil(R)$ is divided. Let $(0, k) \in R$, and $(a, b) \in R \setminus Nil(R)$. Then $a \neq 0$, and hence $(0, k) = (a, b)(0, k/a)$. Observe that $k/a \in K$ because $F \subseteq K$. Thus $R \in \mathcal{H}$. R is not a ZPUI ring by Lemma 2.11. Since $R/Nil(R) \cong A$ is a ZPUI domain, R is a ϕ-ZPUI ring by Theorem 2.3. The Krull dimension of R is n by Proposition 2.12(1). □

In the following theorem, we show that a ϕ-ZPUI ring is a pullback of a ZPUI domain. A good paper for pullback is the article by Fontana in [13].

Theorem 2.14. *Let $R \in \mathcal{H}$. Then R is a ϕ-ZPUI ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is a ZPUI ring that is a subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Suppose $\phi(R)$ is ring-isomorphic to a ring A obtained from the given diagram. Then $A \in \mathcal{H}$ and $Nil(A) = Z(A) = M$. Since A/M is a ZPUI domain, A is a ϕ-ZPUI ring by Theorem 2.3, and thus R is a ϕ-ZPUI ring.

Conversely, suppose that R is a ϕ-ZPUI ring. Then, letting $T = R_{Nil(R)}$, $M = Nil(R_{Nil(R)})$, and $A = \phi(R)$ yields the desired pullback diagram. □

3. nonnil-ZPUI Rings and nonnil-ZPI rings

We start with the following result.

Theorem 3.1. *Let $R \in \mathcal{H}$ be a nonnil-ZPUI ring. Then R is a ϕ-ZPUI ring, and hence all the following statements hold:*

1. *$R/Nil(R)$ is a ZPUI domain.*
2. *Every nonnil proper ideal of R can be written as a product of prime ideals of R and a finitely generated ideal of R.*
3. *Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$.*
4. *R is a nonnil-strongly discrete nonnil-h-local ϕ-Prüfer ring.*
5. *R is a nonnil-strongly discrete nonnil-h-local Prüfer ring.*

Proof. Let L be a nonnil proper ideal ideal of $\phi(R)$. Then $L = \phi(I)$ for some nonnil proper ideal I of R. Since $I = JP_1P_2 \cdots P_n$, where J is an invertible ideal of R and $P_1, P_2, \ldots, P_n$ are prime ideals of R, it follows that $L = \phi(I) = \phi(J)\phi(P_1) \cdots \phi(P_n)$, where $\phi(J)$ is an invertible ideal of $\phi(R)$ and $\phi(P_1), \phi(P_2), \ldots, \phi(P_n)$ are prime ideals of $\phi(R)$. Thus R is a ϕ-ZPUI ring. Now statement **(1)** is clear by Theorem 2.3, and the statements **(2)**, **(3)**, and **(4)** are clear from Theorem 2.9. For statement **(5)**, by [2] just observe that R is a Prüfer ring because R is a ϕ-Prüfer ring. □

In the following result, we show that if $R \in \mathcal{H}$ is a ϕ-ZPUI ring, then R does not need to be a nonnil-ZPUI ring. In particular, we show that if $R \in \mathcal{H}$ satisfies any of the five statements in Theorem 3.1, then R does not need to be a nonnil-ZPUI ring.

Theorem 3.2. *Let A be a ZPUI domain that is not a Dedekind domain with Krull dimension $n \geq 1$ and quotient field K. Then $R = A(+)K/A \in \mathcal{H}$ is a ϕ-ZPUI ring with Krull dimension n which is not a nonnil-ZPUI ring.*

Proof. Since $Nil(R) = \{0\}(+)K/A$, by a similar calculation as in the proof of Theorem 2.13, we conclude that $Nil(R)$ is divided, and thus $R \in \mathcal{H}$. Since $R/Nil(R) \cong A$ is a ZPUI domain, R is a ϕ-ZPUI ring by Theorem 2.3 and the Krull dimension of R is n by Proposition 2.12(1). Since every nonunit element of R is a zerodivisor of R by Proposition 2.12(2), we conclude that $T(R) = R$, and thus R is the only invertible ideal of R. Suppose that R is a nonnnil-ZPUI ring. Then every nonnil proper ideal of R is a finite product of prime ideals of R, and hence every proper ideal of the integral domain $R/Nil(R) \cong A$ is a finite product of prime ideals of $R/Nil(R)$. Thus $R/Nil(R) \cong A$ is a Dedekind domain, a contradiction. Hence R is not a nonnil-ZPUI ring. □

Recall from [16] that a ring R is called a *ZPI-ring* if every nonzero proper ideal of R is uniquely a product of prime ideals of R, and R is called a *general ZPI-ring* if every nonzero proper ideal of R is a product of prime ideals of R. In [4], it is said that a ring $R \in \mathcal{H}$ is a *nonnil-ZPI-ring* if every nonnil proper ideal of R is uniquely a product of (nonnil) prime ideals of R, and it is said that R is a *general nonnil-ZPI-ring* if every nonnil proper ideal of R is a product of (nonnil) prime ideals of R. A ring $R \in \mathcal{H}$ is called a *ϕ-Dedekind ring* as in [4] if every nonnil ideal of R is invertible. A ring $R \in \mathcal{H}$ is called a *nonnil-Noetherian ring* as in [9] if every nonnil ideal of R is finitely generated. We recall the following two results from [4].

Proposition 3.3. *(*[4, Corollary 2.17]*). Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

1. *R is a ϕ-Dedekind ring;*
2. *R is a nonnil-ZPI-ring;*
3. *R is a general nonnil-ZPI-ring.*

Proposition 3.4. *(*[4, Proposition 2.11]*). Let $R \in \mathcal{H}$ be a nonnil-Noetherian ring. Then R is a ϕ-Dedekind ring if and only if R is a ϕ-Prüfer ring.*

Combining Propositions 3.3 and 3.4 with Theorem 2.9, we arrive at the following result.

Corollary 3.5. *Let $R \in \mathcal{H}$ be a nonnil-Noetherian ring. Then the following statements are equivalent:*

1. *R is a ϕ-ZPUI ring;*
2. *R is a nonnil-ZPUI ring;*
3. *R is a nonnil-ZPI ring;*
4. *R is a general nonnil-ZPI ring;*
5. *R is a nonnil-strongly discrete nonnil-h-local ϕ-Prüfer ring;*
6. *R is a nonnil-strongly discretere nonnil-h-local ϕ-Dedekind ring.*

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