FACTORING NONNIL IDEALS INTO PRIME AND INVERTIBLE IDEALS

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ABSTRACT. For a commutative ring $R$, let $\text{Nil}(R)$ be the set of all nilpotent elements of $R$, $Z(R)$ be the set of all zero divisors of $R$, $T(R)$ be the total quotient ring of $R$, and $\mathcal{H} = \{R \mid R$ is a commutative ring and $\text{Nil}(R)$ is a divided prime ideal of $R\}$. For a ring $R \in \mathcal{H}$, let $\phi : T(R) \rightarrow R_{\text{Nil}(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and $b \in R \setminus Z(R)$. A ring $R$ is called a ZPUI ring if every proper ideal of $R$ can be written as a finite product of invertible and prime ideals of $R$. In this paper, we give a generalization of the concept of ZPUI domains which was extensively studied by Olberding to the context of rings that are in the class $\mathcal{H}$. Let $R \in \mathcal{H}$. If every non-nil ideal of $R$ can be written as a finite product of invertible and prime ideals of $R$, then $R$ is called a non-nil-ZPUI ring; if every non-nil ideal of $(R)$ can be written as a finite product of invertible and prime ideals of $\phi(R)$, then $R$ is called a non-nil-$\phi$-ZPUI ring. We show that the theory of $\phi$-ZPUI rings resembles that of ZPUI domains.

1. INTRODUCTION

We assume throughout that all rings are commutative with $1 \neq 0$. Let $R$ be a ring. Then $T(R)$ denotes the total quotient ring of $R$, $Z(R)$ denotes the set of zero divisors of $R$, and $\text{Nil}(R)$ denotes the set of nilpotent elements of $R$. The elements in $R \setminus Z(R)$ are referred to as regular elements and an ideal $I$ is said to be regular if it contains at least one regular element. For a nonzero ideal $I$, regular or not, we let $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$. An ideal $I$ of a ring $R$ is called invertible if $II^{-1} = R$.

We start by recalling some background materials. Recall from [12] and [5] that a prime ideal $P$ of $R$ is called a divided prime if $P \subseteq (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of $R$. In [4], [6], [7], [9], and [8], the author investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring and $\text{Nil}(R)$ is a divided prime ideal of $R\}$. Also, D.F. Anderson, Tom Lucas, and the author made further investigation on the class $\mathcal{H}$ in [2], [3], and most recently in [10]. In this paper, we give a generalization of the concept of factorization of ideals of an integral domain into a finite product of invertible and prime ideals which was extensively studied by Olberding [20] to the context of rings that are in the class $\mathcal{H}$. Observe that if $R$ is an integral domain, then $R \in \mathcal{H}$. An ideal $I$ of a ring $R$ is said to be a non-nil ideal if $I \nsubseteq \text{Nil}(R)$. For each $R \in \mathcal{H}$, the map $\phi : T(R) \rightarrow R_{\text{Nil}(R)}$ defined by $\phi(a/b) = a/b$ for each $a \in R$ and $b \in R \setminus Z(R)$ was introduced by the author in [4]. The map $\phi$ is a ring homomorphism from $T(R)$ into $R_{\text{Nil}(R)}$ and $\phi$ restricted to $R$ is a ring homomorphism from $R$ into $\text{Nil}(R)$.

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$R_{\text{Nil}(R)}$ given by $\phi(x) = x/1$ for each $x \in R$. Note that $T(\phi(R)) = R_{\text{Nil}(R)}$. Let $R \in \mathcal{H}$. Then $R$ is said to be a $\phi$-ZPUI ring, if each nonnil ideal $I$ of $\phi(R)$ can be written as $I = JP_1P_2\cdots P_n$, where $J$ is an invertible ideal of $\phi(R)$ and $P_1, P_2, \ldots, P_n$ are prime ideals of $\phi(R)$. If every nonnil ideal $I$ of $R$ can be written as $I = JP_1P_2\cdots P_n$, where $J$ is an invertible ideal of $R$ and $P_1, P_2, \ldots, P_n$ are prime ideals of $R$, then $R$ is said to be a nonnil-$\phi$-ZPUI ring. Commutative $\phi$-ZPUI rings that are in $\mathcal{H}$ are characterized in Theorem 2.9.

Examples of $\phi$-ZPUI rings that are not ZPUI rings are constructed in Theorem 2.13. It is shown in Theorem 2.14 that a $\phi$-ZPUI ring is the pullback of a ZPUI domain. It is shown in Theorem 3.1 that a nonnil-$\phi$-ZPUI ring is a $\phi$-ZPUI ring. Examples of $\phi$-ZPUI rings that are not nonnil-$\phi$-ZPUI rings are constructed in Theorem 3.2.

If $\text{Nil}(R)$ is divided, then it is also the nilradical of $T(R)$ and the kernel of the map $\phi$ is also a common ideal of $R$ and $T(R)$. Other useful features of each ring $R \in \mathcal{H}$ (see [4]) include the following: (i) $\phi(R) \in \mathcal{H}$, (ii) $T(\phi(R)) = R_{\text{Nil}(R)}$ is quasilocal with maximal ideal $\text{Nil}(\phi(R))$, (iii) $\phi(R)$ is naturally isomorphic to $R/\text{Ker}(\phi)$, (iv) $\text{Nil}(R_{\text{Nil}(R)}) = \phi(\text{Nil}(R)) = \text{Nil}(\phi(R)) = Z(\phi(R))$, and (v) $R_{\text{Nil}(R)}/\text{Nil}(\phi(R)) = T(\phi(R))/\text{Nil}(\phi(R))$ is the quotient field of $\phi(R)/\text{Nil}(\phi(R))$. If $I$ is a nonnil ideal of a ring $R \in \mathcal{H}$, then observe that $\text{Nil}(R) \subseteq I$.

Throughout the paper we will use the technique of idealization of a module to construct examples. Recall that for an $R$-module $B$, the idealization of $B$ over $R$ is the ring formed from $R \times B$ by defining addition and multiplication as $(r, a) + (s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa)$, respectively. A standard notation for the “idealized ring” is $R(+)B$. See [17], [18] and [19] for basic properties of these rings.

2. $\phi$-ZPUI RINGS

We recall the following two lemmas from [2].

**Lemma 2.1.** ([2, Lemma 2.3]) Let $R \in \mathcal{H}$ with $\text{Nil}(R) = Z(R)$, and let $I$ be an ideal of $R$. Then $I$ is an invertible ideal of $R$ if and only if $I/\text{Nil}(R)$ is an invertible ideal of $R/\text{Nil}(R)$.

**Lemma 2.2.** ([2, Lemma 2.5]) Let $R \in \mathcal{H}$ and let $P$ be a prime ideal of $R$. Then $R/P$ is ring-isomorphic to $\phi(R)/\phi(P)$. In particular, $R/\text{Nil}(R)$ is ring-isomorphic to $\phi(R)/\text{Nil}(\phi(R))$.

**Theorem 2.3.** Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-ZPUI ring if and only if $R/\text{Nil}(R)$ is a ZPUI domain.

**Proof.** Suppose that $R$ is a $\phi$-ZPUI ring. Set $D = \phi(R)/\text{Nil}(\phi(R))$, and let $L$ be a nonzero ideal of $D$. Then $L = I/\text{Nil}(\phi(R))$ for some nonnil ideal $I$ of $\phi(R)$. Thus $I = JP_1P_2\cdots P_n$, where $J$ is an invertible ideal of $\phi(R)$ and $P_1, P_2, \ldots, P_n$ are prime ideals of $\phi(R)$. Since $\text{Nil}(\phi(R)) = Z(\phi(R))$, we conclude that $J/\text{Nil}(\phi(R))$ is an invertible ideal of $D$ by Lemma 2.1. Thus $L = I/\text{Nil}(\phi(R)) = (J/\text{Nil}(\phi(R)))(P_1/\text{Nil}(\phi(R)))\cdots(P_n/\text{Nil}(\phi(R)))$, and hence $D$ is a ZPUI domain. Since $D$ is ring-isomorphic to $R/\text{Nil}(R)$ by Lemma 2.2, we conclude that $R/\text{Nil}(R)$ is a ZPUI domain.

Conversely, suppose that $R/\text{Nil}(R)$ is a ZPUI domain. Then $D = \phi(R)/\text{Nil}(\phi(R))$ is a ZPUI domain by Lemma 2.2. Let $I$ be a nonnil ideal of $\phi(R)$. Since $\phi(R) \in \mathcal{H}$, $I/\text{Nil}(\phi(R))$ is a nonzero ideal of $D$. Thus $I/\text{Nil}(\phi(R)) =$
Lemma 2.4. Let $R \in \mathcal{H}$ and let $I$ be a nonnil ideal of $R$. Then $I$ is a finitely generated ideal of $R$ if and only if $I/\text{Nil}(R)$ is a finitely generated ideal of $R/\text{Nil}(R)$.

Proof. (the proof is similar to the proof of [9, Theorem 2.2].) Suppose that $I$ is a nonnil finitely generated ideal of $R$. Since $\text{Nil}(R) \subseteq I$, it is clear that $I/\text{Nil}(R)$ is a finitely generated ideal of $R/\text{Nil}(R)$. Conversely, suppose that $J = I/\text{Nil}(R)$ is a finitely generated ideal of $R/\text{Nil}(R)$. Then $J = (i_1 + \text{Nil}(R), \ldots, i_n + \text{Nil}(R))$ for some $i_m$’s in $I$. Since $\text{Nil}(R)$ is divided, we may assume that all the $i_m$’s are nonnilpotent elements of $R$, and thus $\text{Nil}(R) \subseteq (i_1)$. Now let $x$ be a nonnilpotent element of $I$. Then $x + \text{Nil}(R) = c_1 i_1 + \ldots + c_n i_n + \text{Nil}(R)$ in $R/\text{Nil}(R)$ for some $c_n$’s in $R$. Hence there is a $w \in \text{Nil}(R)$ such that $x + w = c_1 i_1 + \ldots + c_n i_n$ in $R$. Since $x \in I \setminus \text{Nil}(R)$, $x | w$ in $R$. Thus $w = xf$ for some $f \in \text{Nil}(R)$. Hence $x + w = x + xf = x(1 + f) = c_1 i_1 + \ldots + c_n i_n$ in $R$. Since $f \in \text{Nil}(R)$, $1 + f$ is a unit of $R$. Thus $x \in (i_1, \ldots, i_n)$, and hence $I$ is a finitely generated ideal of $R$. \qed

Recall from [19] that a ring $R$ is called a Prüfer ring if every finitely generated regular ideal of $R$ is invertible. A Prüfer domain $R$ is called a strongly discrete Prüfer domain as in [21] and [20] if $R$ has no nonzero prime ideals $P$ such that $P^2 = P$. A ring $R \in \mathcal{H}$ is said to be a φ-Prüfer ring as in [2] if $\phi(R)$ is a Prüfer ring. We call a ring $R \in \mathcal{H}$ a nonnil-strongly discrete ring if $R$ has no nonnil prime ideal $P$ such that $P^2 = P$. An integral domain $R$ is called h-local as in [21] if each nonzero ideal of $R$ is contained in at most finitely many maximal ideals of $R$ and each nonzero prime ideal $P$ of $R$ is contained in a unique maximal ideal of $R$. A ring $R \in \mathcal{H}$ is said to be nonnil-h-local if each nonnil ideal of $R$ is contained in at most finitely many maximal ideals of $R$ and each nonnil prime ideal $P$ of $R$ is contained in a unique maximal ideal of $R$.

The reader can easily verify the following two lemmas.

Lemma 2.5. Let $R \in \mathcal{H}$. Then $R$ is a nonnil-h-local ring if and only if $R/\text{Nil}(R)$ is an h-local domain.

Lemma 2.6. Let $R \in \mathcal{H}$. Then $R$ is a nonnil-strongly discrete Prüfer ring if and only if $R/\text{Nil}(R)$ is a strongly discrete Prüfer domain.

We recall the following result from [2].

Proposition 2.7. ([2]) Let $R \in \mathcal{H}$. Then $R$ is a φ-Prüfer ring if and only if $R/\text{Nil}(R)$ is a Prüfer domain.

Combining Lemmas 2.5 and 2.6 with Proposition 2.7, we arrive at the following result.

Proposition 2.8. Let $R \in \mathcal{H}$. Then $R$ is a nonnil-strongly discrete nonnil-h-local φ-Prüfer ring if and only if $R/\text{Nil}(R)$ is a strongly discrete h-local Prüfer domain.
Since the class of integral domains is a subset of $\mathcal{H}$, the following result is a generalization of [20, Theorem 2.3].

**Theorem 2.9.** Let $R \in \mathcal{H}$. Then the following statements are equivalent:

1. $R$ is a $\phi$-ZPUI ring;
2. Every nonnil proper ideal of $R$ can be written as a product of prime ideals of $R$ and a finitely generated ideal of $R$;
3. Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$;
4. $R$ is a nonnil-strongly discrete nonnil-h-local $\phi$-Prüfer ring.

**Proof.** Set $D = R/\text{Nil}(R)$. (1) $\Rightarrow$ (2). Since $R$ is a $\phi$-ZPUI ring, $D$ is a ZPUI domain by Theorem 2.3. Let $I$ be a nonnil proper ideal of $R$. Then by [20, Theorem 2.3] we have $I/\text{Nil}(R) = J/\text{Nil}(R)P_1/\text{Nil}(R)\cdots P_n/\text{Nil}(R)$, where $J$ is a (nonnil) finitely generated ideal of $R$ (by Lemma 2.4) and $P_1, P_2, \ldots, P_n$ are prime ideals of $R$. Since $\text{Nil}(R)$ is divided, it is easily verified that $I = JP_1\cdots P_n$.

(2) $\Rightarrow$ (3). Let $L$ be a nonnil proper ideal of $\phi(R)$. Then $L = \phi(I)$ for some nonnil proper ideal $I$ of $R$. Since $I = JP_1\cdots P_n$, where $J$ is a (nonnil) finitely generated ideal of $R$ and $P_1, P_2, \ldots, P_n$ are prime ideals of $R$, it is easily verified that $L = \phi(I) = \phi(J)\phi(P_1)\cdots \phi(P_n)$, where $\phi(J)$ is a finitely generated ideal of $\phi(R)$ and $\phi(P_1), \ldots, \phi(P_n)$ are (nonnil) prime ideals of $\phi(R)$.

(3) $\Rightarrow$ (4). Let $F = \phi(R)/\text{Nil}(\phi(R))$. Then every nonzero ideal of $F$ can be written as a product of prime ideals of $F$ and a finitely generated ideal of $F$, and thus $F$ is a strongly discrete h-local Prüfer domain by [20, Theorem 2.3]. Since $F$ is ring-isomorphic to $D$, we conclude that $D$ is a strongly discrete h-local Prüfer domain, and hence $R$ is a nonnil-strongly discrete nonnil-h-local $\phi$-Prüfer ring by Proposition 2.8.

(4) $\Rightarrow$ (1). Since $R$ is a nonnil-strongly discrete nonnil-h-local $\phi$-Prüfer ring, we conclude that $D = R/\text{Nil}(R)$ is a strongly discrete h-local Prüfer domain by Proposition 2.8. Thus $D$ is a ZPUI domain by [20, Theorem 2.3], and hence $R$ is a $\phi$-ZPUI ring by Theorem 2.3.

Let $R \in \mathcal{H}$ such that $Z(R) = \text{Nil}(R)$. Then $\phi(R) = R$, and hence $R$ is a $\phi$-ZPUI ring if and only if $R$ is a nonnil-ZPUI ring. We state this connection in the following corollary.

**Corollary 2.10.** Let $R \in \mathcal{H}$ such that $\text{Nil}(R) = Z(R)$. The following statements are equivalent:

1. $R$ is a nonnil-ZPUI ring;
2. $R$ is a $\phi$-ZPUI ring;
3. Every nonnil proper ideal of $R$ can be written as a product of prime ideals of $R$ and a finitely generated ideal of $R$;
4. $R$ is a nonnil-strongly discrete nonnil-h-local Prüfer ring.

Recall that a special primary ring is a quasilocal commutative ring $R$ with maximal ideal $M$ such that every proper ideal of $R$ is a power of $M$. We state the following useful lemma.

**Lemma 2.11.** (see [20, Lemma 3.2 and Theorem 3.3]). Let $R \in \mathcal{H}$. Then $R$ is a ZPUI ring if and only if $R$ is either a strongly discrete h-local Prüfer domain or a special primary ring.
Proof. Suppose that $R$ is a ZPUI ring. First observe that if a ring $A \cong A_1 \oplus \cdots \oplus A_n$ (where each $A_i$ is a ring with $1 \neq 0$) and $n \geq 2$, then $\text{Nil}(A)$ is never divided, and hence $A \not\in \mathcal{H}$. Now since $R$ is a ZPUI ring, by [20, Theorem 3.3] we have $R \cong D_1 \oplus \cdots \oplus D_n$, where each $D_i$ is either a strongly discrete h-local Prüfer domain or a special primary ring. Since $\text{Nil}(R)$ is divided, by the observation we just stated, we conclude that $n = 1$, and thus $R$ is either a strongly discrete h-local Prüfer domain or a special primary ring. The converse is clear by [20, Theorem 3.3].

Our non-domain examples of $\phi$-ZPUI rings that are not ZPUI rings are provided by the idealization construction $R(+)B$ arising from a ring $R$ and an $R$-module $B$ as in [19, Chapter VI]. We recall this construction. Let $R(+)B = R \times B$, and define:

1. $(r, b) + (s, c) = (r + s, b + c)$.
2. $(r, b)(s, c) = (rs, sb + rc)$.

Under these definitions $R(+)B$ becomes a commutative ring with identity. We recall the following two facts.

Proposition 2.12. Let $R$ be a ring, $B$ be an $R$-module, and $Z(B)$ be the set of zero-divisors on $B$. Then:

1. ([19, Theorem 25.1]) The ideal $J$ of $R(+)B$ is prime (respectively, maximal) if and only if $J = P(+)B$, where $P$ is a prime (respectively, maximal) ideal of $R$, and hence the Krull dimension of $R$ is equal to the Krull dimension of $R(+)B$;
2. ([19, Theorem 25.3]) $(r, b) \in Z(R(+)B)$ if and only if $r \in Z(R) \cup Z(B)$.

Olberding in [20, Corollary 2.4] showed that for each $n \geq 1$, there exists a ZPUI domain with Krull dimension $n$. A Dedekind domain is a trivial example of a ZPUI domain.

Theorem 2.13. Let $A$ be a ZPUI domain (i.e. $A$ is a strongly discrete h-local Prüfer domain by [20, Theorem 2.3]) with Krull dimension $n \geq 1$ and quotient field $F$, and let $K$ be an extension ring of $F$ (i.e. $K$ is a ring and $F \subseteq K$). Then $R = A(+)K \in \mathcal{H}$ is a $\phi$-ZPUI ring with Krull dimension $n$ that is not a ZPUI ring.

Proof. It is easy to see that $\text{Nil}(R) = \{0\}(+)K$. We show that $\text{Nil}(R)$ is divided. Let $(0, k) \in R$, and $(a, b) \in R \setminus \text{Nil}(R)$. Then $a \neq 0$, and hence $(0, k) = (a, b)(0, k/a)$. Observe that $k/a \in K$ because $F \subseteq K$. Thus $R \in \mathcal{H}$. $R$ is not a ZPUI ring by Lemma 2.11. Since $R/\text{Nil}(R) \cong A$ is a ZPUI domain, $R$ is a $\phi$-ZPUI ring by Theorem 2.3. The Krull dimension of $R$ is $n$ by Proposition 2.12(1).

In the following theorem, we show that a $\phi$-ZPUI ring is a pullback of a ZPUI domain. A good paper for pullback is the article by Fontana in [13].

Theorem 2.14. Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-ZPUI ring if and only if $\phi(R)$ is ring-isomorphic to a ring $A$ obtained from the following pullback diagram:

$$
\begin{align*}
A & \longrightarrow A/M \\
\downarrow & \downarrow \\
T & \longrightarrow T/M
\end{align*}
$$
where $T$ is a zero-dimensional quasilocal ring with maximal ideal $M$, $A/M$ is a ZPUI ring that is a subring of $T/M$, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Suppose $\phi(R)$ is ring-isomorphic to a ring $A$ obtained from the given diagram. Then $A \in \mathcal{H}$ and $\Nil(A) = Z(A) = M$. Since $A/M$ is a ZPUI domain, $A$ is a $\phi$-ZPUI ring by Theorem 2.3, and thus $R$ is a $\phi$-ZPUI ring.

Conversely, suppose that $R$ is a $\phi$-ZPUI ring. Then, letting $T = R_{\Nil(R)}$, $M = \Nil(R_{\Nil(R)})$, and $A = \phi(R)$ yields the desired pullback diagram. \qed

3. Nonnil-ZPUI Rings and Nonnil-ZPI Rings

We start with the following result.

Theorem 3.1. Let $R \in \mathcal{H}$ be a nonnil-ZPUI ring. Then $R$ is a $\phi$-ZPUI ring, and hence all the following statements hold:

1. $R/\Nil(R)$ is a ZPUI domain.
2. Every nonnil proper ideal of $R$ can be written as a product of prime ideals of $R$ and a finitely generated ideal of $R$.
3. Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$.
4. $R$ is a nonnil-strongly discrete nonnil-h-local $\phi$-Prüfer ring.
5. $R$ is a nonnil-strongly discrete nonnil-h-local Prufer ring.

Proof. Let $L$ be a nonnil proper ideal ideal of $\phi(R)$. Then $L = \phi(I)$ for some nonnil proper ideal $I$ of $R$. Since $I = JP_1P_2\cdots P_n$, where $J$ is an invertible ideal of $R$ and $P_1, P_2, \ldots, P_n$ are prime ideals of $R$, it follows that $L = \phi(I) = \phi(J)\phi(P_1)\cdots\phi(P_n)$, where $\phi(J)$ is an invertible ideal of $\phi(R)$ and $\phi(P_1), \phi(P_2), \ldots, \phi(P_n)$ are prime ideals of $\phi(R)$. Thus $R$ is a $\phi$-ZPUI ring. Now statement (1) is clear by Theorem 2.3, and the statements (2), (3), and (4) are clear from Theorem 2.9. For statement (5), by [2] just observe that $R$ is a Prüfer ring because $R$ is a $\phi$-Prüfer ring. \qed

In the following result, we show that if $R \in \mathcal{H}$ is a $\phi$-ZPUI ring, then $R$ does not need to be a nonnil-ZPUI ring. In particular, we show that if $R \in \mathcal{H}$ satisfies any of the five statements in Theorem 3.1, then $R$ does not need to be a nonnil-ZPUI ring.

Theorem 3.2. Let $A$ be a ZPUI domain that is not a Dedekind domain with Krull dimension $n \geq 1$ and quotient field $K$. Then $R = A(+K)/A \in \mathcal{H}$ is a $\phi$-ZPUI ring with Krull dimension $n$ which is not a nonnil-ZPUI ring.

Proof. Since $\Nil(R) = \{0\}(+K)/A$, by a similar calculation as in the proof of Theorem 2.13, we conclude that $\Nil(R)$ is divided, and thus $R \in \mathcal{H}$. Since $R/\Nil(R) \cong A$ is a ZPUI domain, $R$ is a $\phi$-ZPUI ring by Theorem 2.3 and the Krull dimension of $R$ is $n$ by Proposition 2.12(1). Since every nonunit element of $R$ is a zerodivisor of $R$ by Proposition 2.12(2), we conclude that $T(R) = R$, and thus $R$ is the only invertible ideal of $R$. Suppose that $R$ is a nonnil-ZPUI ring. Then every nonnil proper ideal of $R$ is a finite product of prime ideals of $R$, and hence every proper ideal of the integral domain $R/\Nil(R) \cong A$ is a finite product of prime ideals of $R/\Nil(R)$. Thus $R/\Nil(R) \cong A$ is a Dedekind domain, a contradiction. Hence $R$ is not a nonnil-ZPUI ring. \qed
Recall from [16] that a ring $R$ is called a ZPI-ring if every nonzero proper ideal of $R$ is uniquely a product of prime ideals of $R$, and $R$ is called a general ZPI-ring if every nonzero proper ideal of $R$ is a product of prime ideals of $R$. In [4], it is said that a ring $R \in \mathcal{H}$ is a nonnil-ZPI-ring if every nonnil proper ideal of $R$ is uniquely a product of (nonnil) prime ideals of $R$, and it is said that $R$ is a general nonnil-ZPI-ring if every nonnil proper ideal of $R$ is a product of (nonnil) prime ideals of $R$. A ring $R \in \mathcal{H}$ is called a nonnil-Dedekind ring as in [4] if every nonnil ideal of $R$ is invertible. A ring $R \in \mathcal{H}$ is called a nonnil-Noetherian ring as in [9] if every nonnil ideal of $R$ is finitely generated. We recall the following two results from [4].

**Proposition 3.3.** ([4, Corollary 2.17]). Let $R \in \mathcal{H}$. Then the following statements are equivalent:

1. $R$ is a $\phi$-Dedekind ring;
2. $R$ is a nonnil-ZPI-ring;
3. $R$ is a general nonnil-ZPI-ring.

**Proposition 3.4.** ([4, Proposition 2.11]). Let $R \in \mathcal{H}$ be a nonnil-Noetherian ring. Then $R$ is a $\phi$-Dedekind ring if and only if $R$ is a $\phi$-Prüfer ring.

Combining Propositions 3.3 and 3.4 with Theorem 2.9, we arrive at the following result.

**Corollary 3.5.** Let $R \in \mathcal{H}$ be a nonnil-Noetherian ring. Then the following statements are equivalent:

1. $R$ is a $\phi$-ZPUI ring;
2. $R$ is a nonnil-ZPUI ring;
3. $R$ is a nonnil-ZPI ring;
4. $R$ is a general nonnil-ZPI ring;
5. $R$ is a nonnil-strongly discrete nonnil-h-local $\phi$-Prüfer ring;
6. $R$ is a nonnil-strongly discretete nonnil-h-local $\phi$-Dedekind ring.

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