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Study Manual
for
Business Placement Test

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Contents

1 Fundamentals of Algebra

The Real Numbers

Definition 1.1 *Natural numbers* $1, 2, 3, ...$

Whole numbers $0, 1, 2, 3, ...$

Integers $..., -3, -2, -1, 0, 1, 2, ...$

Rational numbers $\frac{p}{q}$, *where* p *and* q *are integers and* $q \neq 0$

Irrational numbers *numbers whose decimal part does not terminate or repeat.*

Real numbers *all rational and all irrational numbers.*

Properties of Operations: For any real numbers a, b, c

1. Closure of addition $a + b$ is a real number.
2. Closure of multiplication $a \cdot b$ is a real number.
3. Commutative of Addition $a + b = b + a$.
4. Commutative of multiplication $a \cdot b = b \cdot a$.
5. Associative of Addition $(a + b) + c = a + (b + c)$.
6. Associative of multiplication $a(b + c) = a \cdot b + a \cdot c$.
7. Identity for Addition $a + 0 = a$.
8. Identity for multiplication $a \cdot 1 = a$.

a.

Order of operation:

Perform operations within the innermost grouping symbols according to steps 1-3 below.

1. Perform operations indicated by exponents (powers).
2. Perform multiplication and division in order from left to right.
3. Perform addition and subtraction in order from left to right.

Example 1.1 *Evaluate* $\dfrac{2^2(13+5)}{4}$

Solution:

$$\frac{2^2(18)}{4} = \frac{4(18)}{4}$$
$$= \frac{72}{4} = 18$$

Example 1.2 *Evaluate* $\dfrac{18-2\cdot 5+3}{15+3(-3)}$

Solution:

$$\frac{18-10+3}{15-9}$$
$$\frac{8+3}{6}$$
$$\frac{11}{6}$$

Exercises

1. Evaluate $(7-3^2)2+4\div 3+3$.
2. Evaluate $\dfrac{5\cdot 6\div 3\cdot 7}{12}$.

3. Evaluate $2[14 - 3(6-1)^2]$.

Percent

percent means the number of hundredths or the number of parts out of 100

Definition 1.2 $p\% = \frac{p}{100}$

- To rewrite a decimal as a percent, move the decimal point two places to the right and add the percent sign $(\%)$.
- To rewrite fraction as a percent, first change the fraction to decimal, then change the decimal to a percent.

Example 1.3 *write the following as a percent value*

1. $0.45 = 45\%$
2. $\dfrac{3}{4} = 0.75 = 75\%$
3. $\dfrac{5}{2} = 2.5 = 250\%$

Remark 1.1 *To rewrite a percent as a decimal, drop the percent symbol and divide the number that remains by 100.*

Example 1.4 *Rewrite the following as a decimal or fraction.*

1. $85\% = \dfrac{85}{100} = 0.85.$
2. $\dfrac{3}{4}\% = \frac{\frac{3}{4}}{100} = \frac{0.75}{100} = 0.0075.$

Example 1.5 *After 2 months on a diet, John's weight dropped from 168 pounds to 147 pounds. By what percent did john's weight drop?*

Solution:

$$\begin{aligned}\text{percent decrease} &= \frac{\text{Amount of decrease}}{\text{Original amount}} \times 100\% \\ &= \frac{168 - 147}{168} \times 100\% \\ &= \frac{21}{168} \times 100\% \\ &= 12.5\%\end{aligned}$$

Example 1.6 *A discount of 25% on the price of a pair of shoes, followed by another discount of 8% on the new price of the shoes , is equivalent to a single discount of what percent of the original price.*

Solution: Suppose the price of shoes cost \$100

$$\begin{aligned}100 \times 25\% &= 25 \\ price &= 100 - 25 \\ price &= 75 \\ final\ price &= \$75 - (8\% \times \$75) \\ &= 75 - 6 \\ &= \$69 \\ &\quad \text{Since the final price is \$31 less than the original price} \\ \frac{31}{100} &= \text{31, the answer is } 31\%\end{aligned}$$

Example 1.7 *In a factory that manufactures light bulbs,* 0.04% *of the bulbs manufactured are defective. It is expected that there will be one defective light bulb in what number of bulbs that are manufactured?*

Solution:

$$0.04\% = \frac{0.04}{100} = \frac{4}{10,000} = \frac{1}{2500}$$

It can be expected that one of every 2500 light bulbs will be defective.

Exercises

1. A high school tennis team is scheduled to play 28 matches. If the team wins 60% of the first 15 matches, how many additional matches must the team win in order to finish the season winning 75% of its scheduled matches?

2. If 25% of x is 12.5, what is 12.5% of $2x$?

Linear Equations

A statement of the form $4(x+2) = x - 7$ is an example of a linear equation because the variable x appears only to first power. To solve an equation means to find the real number x that satisfies the equation. These are called solutions or roots of the equation.

To solve any linear equation collect all terms with the variable on one side of the equation, and the constant on the other side.

Example 1.8 *Solve* $4(x+2) = x - 7$.

Solution: The first step is to eliminate the parentheses.

$$\begin{aligned} 4(x+2) &= x-7 \\ 4x+8 &= x-7 \\ 4x-x &= -7-8 \quad \text{Collect terms with } x \text{ on one side and the constants on the other side} \\ 3x &= -15 \\ x &= \frac{-15}{3} \quad \text{Divide by the coefficient of } x \\ x &= -5 \quad \text{This is the solution.} \end{aligned}$$

We can show that this is the correct solution by substituting $x = -5$ in the original equation.

$4(-5+2) = -5 - 7 \to -12 = -12.$

Example 1.9 *Solve* $2(x-3) + 5 = -3(x+4)$.

Solution:

$$\begin{aligned}
2(x-3)+5 &= -3(x+4) \\
2x-6+5 &= -3x-12 \\
2x+3x &= -12+6-5 \\
5x &= -11 \\
x &= \frac{-11}{5}
\end{aligned}$$

Example 1.10 *Solve* $\frac{7}{2}(x-4) = \frac{5}{2}x - 4.$

Solution:

$$\begin{aligned}
\frac{7}{2}(x-4) &= \frac{5}{2}x-4 \\
\frac{7}{2}x - \frac{7}{2}(4) &= \frac{5}{2}x-4 \\
\frac{7}{2}x - 14 &= \frac{5}{2}x-4 \\
\frac{7}{2}x - \frac{5}{2}x &= 14-4 \\
\frac{2}{2}x &= 10 \\
x &= 10
\end{aligned}$$

Exercises

Solve the following linear equations:

1. $2(x+1) = 11$

2. $-3(x+4) = 2x - 7$

3. $\frac{4}{3}(x+8) = \frac{3}{2}(2x+2)$

4. $x + 2(\frac{1}{6}x - 3) = \frac{6}{5}x + 16$

Problem Solving in Linear equations

One of the main reasons for learning mathematics is to be able to use it to solve application problems in business and science. In this section we will explore the solution of problems that are expressed in words. The task will be to translate the English sentences of a problem into suitable mathematical language and to develop an equation we can solve.

Example 1.11 *Find a number such that two-thirds of the number increased by 3 is 15.*

Solution: Let x be the number. The translation of the English statement *two-thirds of the number increased by 3 is 15* will be $\frac{2}{3}x+3=15$. Now we only need to solve this linear equation.

$$\begin{aligned}\frac{2}{3}x+3 &= 15\\ \frac{2}{3}x &= 15-3\\ \frac{2}{3}x &= 12\\ x &= \frac{12}{(\frac{2}{3})}=18.\end{aligned}$$

Therefore, the number is 18.

Example 1.12 *Ahmad has a base salary of \$250 per week. In addition, he receives a commission of 10% of his salary. Last week his total earnings were \$560. What were the total sales for the week?*

Solution: Let x represents the total sale for the week. Then

$$\begin{aligned}\text{Salary +Commission} &= \text{Total Earnings}\\ 250+0.1x &= 560\\ 0.1x &= 560-250\\ 0.1x &= 310\\ x &= \frac{310}{0.1}\\ x &= \$3100\end{aligned}$$

Ahmad's total sales for the week were \$3100.

Example 1.13 *Muna received \$52,000 profit from the sale of a land. She invested part at 5% interest, and the rest at 4% interest. She earned a total of \$2290 interest per year. How much did she invest at 5%?*

Solution: Let x represents the total amount invested at 5% interest. Then 52,000$-x$ will be the amount invested at 4% interest rate. Since she earned \$2290 interest per year, then

$$\begin{aligned} 0.05x + (52000 - x)0.04 &= 2290 \\ 0.05x + 52000(0.04) - 0.04x &= 2290 \\ 0.01x + 2080 &= 2290 \\ 0.01x &= 2290 - 2080 \\ 0.01x &= 210 \\ x &= \frac{210}{0.01} = 21000 \end{aligned}$$

Therefore, she invested \$21000 at 5% interest.

Exercises

1. Find a number such that two-fifths of the number increased by 4 is 18.

2. Salem invests \$20,000 received from an insurance settlement in two ways: Some at 6%, and some at 4%. Altogether, he makes \$1040 per year interest. How much is invested at 4%?

3. Maria has \$169 in ones, fives, and tens. She has twice as many one-dollar bills as she has five-dollar bills, and five more ten-dollar bills than five-dollar bills. How many of each type bill does she have?

Linear Inequalities

An inequality is a statement that has one mathematical expression is greater than or equal (or less than or equal) another. Inequalities are very important in applications. For example, a company wants revenue to be greater than costs and must use no more than the total amount of capital or labor available.

The following properties are the basic algebraic tools for working with inequalities.

Properties of Inequality

For any numbers $a,\ b$, and c,
1. if $a < b$, then $a + c < b + c$
2. if $a < b$, and if $c > 0$, then $ac < bc$
3. if $a < b$, and if $c < 0$, then $ac > bc$.

Throughout this section, definition are given only for $<$; but they are equally valid for $>$, $\leq$, or $\geq$.

Example 1.14 *Solve the following inequalities:*

1. $3x + 3 > 12$
2. $-4x - 3 \leq 7 + 3x$
3. $-3 < 5 + 7m < 10$

Solution
1. First add -3 to both sides
$$\begin{aligned} 3x + 3 - 3 &> 12 - 3 \\ 3x &> 9 \end{aligned}$$
Now multiply both sides by $1/3$
$$\begin{aligned} (1/3)3x &> 9(1/3) \\ x &> 3 \end{aligned}$$
Hence, the solution is $x > 3$. In interval notation we can write the solution as $(3, \infty)$.

2.
$$\begin{aligned} -4x - 3 &\leq 7 + 3x \\ -4x - 3x &\leq 7 + 3 \\ -7x &\leq 10 \\ x &\geq -\frac{10}{7} \end{aligned}$$
The solution is $x \geq -\frac{10}{7}$. Note that we changed the inequality above after multiplying by the negative number $(-1/7)$.

3. The inequality $-3 \leq 5 + 7m < 10$ says that $5 + 7m$ is between -3 and 10.

$$\begin{aligned} -3 &\leq 5 + 7m < 10 \\ -3 - 5 &\leq 7m < 10 - 5 \\ -8 &\leq 7m < 5 \\ -\frac{8}{7} &\leq m < \frac{5}{7}. \end{aligned}$$

The solution is $-\frac{8}{7} \leq m < \frac{5}{7}$. In interval notation the solution is $[-\frac{8}{7}, \frac{5}{7})$.

Exercises

1. $3x + 7 < 5x - 4$.

2. $7x - 5 \geq 6x + 5$.

3. $-5x + 6 \leq 3x + 2 \leq 2x - 5$.

Absolute Value

What do the numbers -4 and 4 have in common? Obviously, they are different numbers and are the coordinates of two distinct points on the number line. However, they are both the same distance from 0, the origin, on the number line.

In another words, -4 is as far to the left of 0 as $+4$ is to the right of 0. We show this fact by using **absolute value notation:**

$$\begin{aligned} |-5| &= 5 \\ |5| &= 5 \end{aligned}$$

Definition 1.3 *For any real number a,*

$$|a| = \left\{ \begin{array}{cc} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{array} \right\}$$

Example 1.15 *Solve for* $|5 - x| = 1$.

Solution:

$$\begin{aligned} 5 - x &= 1 \quad \text{or} \quad -(5 - x) = 1 \\ -x &= -4 \quad \text{or} \quad -5 + x = 1 \\ x &= 4 \quad \text{or} \quad x = 6 \end{aligned}$$

Check: $|5 - 4| = |1| = 1; \quad |5 - 6| = |-1| = 1.$

Example 1.16 *Solve for* $|3x - 5| = 4$.

Solution:

$$\begin{aligned} 3x - 5 &= 4 \quad \text{or} \quad -(3x - 5) = 4 \\ 3x &= 9 \quad \text{or} \quad -3x + 5 = 4 \\ x &= 3 \quad \text{or} \quad -3x = -1 \\ x &= 3 \quad \text{or} \quad x = \frac{1}{3} \end{aligned}$$

Example 1.17 $|x + 7| > 3$

Solution:

$$\begin{aligned} x + 7 &> 3 \quad \text{or} \quad -(x + 7) > 3 \\ x &> -4 \quad \text{or} \quad -x - 7 > 3 \\ x &> -4 \quad \text{or} \quad -x > 10 \\ x &> -4 \quad \text{or} \quad x < -10 \end{aligned}$$

Example 1.18 *Solve* $|x - 5| \geq 2$

Solution:

$$\begin{aligned} -2 &\geq x-5 \geq 2 \\ -2+5 &\geq x-5+5 \geq 2+5 \\ 3 &\geq x \geq 7 \\ x &\in [3,7] \end{aligned}$$

Exercises

Solve the following inequalities:

1. $|x-12| > 6$.

2. $|x-8| \leq -4$.

3. $|7x+4| \leq 18$.

4. $\left|\frac{2}{3}x+\frac{3}{4}\right| > 4$.

Integral Exponents

Much of mathematical notation can be viewed as efficient abbreviations of lengthier statements. For example:

$$2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

This illustration make use of a positive integral exponent. In this section we shall explore the use of integrals as exponents.

Definition 1.4 *If a is a real number and n is a natural number, then* $a^n = a \times a \times a....... \times a,\ n$ *times.*

Definition 1.5 *If a is nonzero, then $a^0 = 1$. If n a natural number, then $a^{-n} = \frac{1}{a^n}$.*

Here are some illustrations of the definition:

$$\begin{aligned} b^1 &= b \\ (a+b)^2 &= a^2 + b^2 + 2ab \\ (-3)^5 &= (-3)(-3)(-3)(-3)(-3) = -243 \end{aligned}$$

Properties of Exponents

Let a and b nonzero real numbers. Let m and n be integers.

1. Product of Powers $(a)^m(a)^n = a^{m+n}$.
2. Quotient of Powers $\frac{a^n}{a^m} = a^{n-m}$.
3. Power of Power $(a^m)^n = a^{mn}$.
4. Power of Product $(ab)^m = a^m b^m$
5. Power of Quotient $(\frac{a}{b})^n = \frac{a^n}{b^n}$

Example 1.19 *Simplify each term.*

1. $3x^2y^{-2}(-2x^3y^{-4})$.

   Solution:

$$\begin{aligned} &(3)(-2)(x^{2+3})(y^{-2+(-4)}) \\ &-6x^5y^{-6} \\ &-\frac{6x^5}{y^6} \end{aligned}$$

2. $\left(\dfrac{-y^7}{2z^{12}y^3}\right)^4$

   Solution:

$$\begin{aligned} \left(\frac{-y^7}{2z^{12}y^3}\right)^4 &= \frac{(-y^7)^4}{(2z^{12}y^3)^4} \\ \frac{y^{28}}{16z^{48}y^{12}} &= \frac{y^{28}}{16z^{48}y^{12}} \\ &= \frac{y^{16}}{16z^{48}} \end{aligned}$$

3. $\left(\dfrac{4a^2b^{-3}}{a^{-1}b^2}\right)^{-2}$

   Solution:

$$\begin{aligned} & (4a^{2-(-1)}b^{-3-2})^{-2} \\ = & (4a^3b^{-5})^{-2} \\ = & (4)^{-2}(a^3)^{-2}(b^{-5})^{-2} \\ = & \frac{b^{10}}{16a^6} \end{aligned}$$

Example 1.20 *Find a value for x to make each statement true.*

1. $2^x.2^3 = 32$

   Solution:

$$\begin{aligned} 2^x.2^3 &= 2^{3+x} = 32 = 2^5 \\ 3+x &= 5 \\ x &= 2 \end{aligned}$$

2. $\dfrac{2^{-3}}{2^x} = 16$

   Solution:

$$\begin{aligned} \frac{2^{-3}}{2^x} &= 2^{-3-x} = 16 = 2^4 \\ -3-x &= 4 \\ x &= -7 \end{aligned}$$

Exercises

1. Simplify the following terms using positive exponents.

1. a. $\left(\dfrac{2x^{-3}y^4}{4x^6y^{-3}}\right)^2$

   b. $-\dfrac{\left(x^2y^4\right)^{-3}}{3x^2y^4}$

   c. $\dfrac{\left(-5a^{-5}b^6\right)^3}{\left(15a^8b^5\right)^4}$

2. Find a value of x to make each statement true.

   a. $2^3 2^{2x} = 2^{16}$

   b. $\dfrac{2^{-x}}{2^3} = 64$

Radicals and Rational Exponents

The positive square root $\sqrt{a}$ is called the principal square root of a. For example the principal square root of 25 is 5.

In general, the **principal nth root** of a real number a is denoted by $\sqrt[n]{a}$ and is equal to $a^{1/n}$.

Definition 1.6 *Let a be a real number and n a positive integer, $n \geq 2$.*

1. If $a > 0$, then $\sqrt[n]{a}$ is the positive real number x such that $x^n = a$.
2. $\sqrt[n]{0} = 0$.
3. If $a < 0$ and n is odd, then $\sqrt[n]{a}$ is the negative real number x such that $x^n = a$.
4. If $a < 0$ and n is even, then $\sqrt[n]{a}$ in not a real number.

The symbol $\sqrt[n]{a}$ is said to be radical. We also define $a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

Example 1.21 *Evaluate:* $(8)^{-2/3} + (-32)^{-2/5}$.

Solution: First rewrite each part using positive exponents. Then apply the definition of

$a^{m/n}$ and add:

$$\begin{aligned} 8^{-2/3}+(-32)^{-2/5} &= \frac{1}{8^{2/3}}+\frac{1}{(-32)^{2/5}} \\ &= \frac{1}{\left(\sqrt[3]{8}\right)^2}+\frac{1}{\left(\sqrt[5]{-32}\right)^2} \\ &= \frac{1}{(2)^2}+\frac{1}{(-2)^2} \\ &= \frac{1}{4}+\frac{1}{4} \\ &= \frac{1}{2} \end{aligned}$$

Example 1.22 *Simplify:* $\left(\dfrac{16x^8}{y^{-4}}\right)^{3/4}$.

Solution:

$$\begin{aligned} \left(\frac{16x^8}{y^{-4}}\right)^{3/4} &= \frac{(16x^8)^{3/4}}{(y^{-4})^{3/4}} \\ &= \frac{(16)^{3/4}(x^8)^{3/4}}{y^{-3}} \\ &= \frac{\sqrt[4]{16^3}x^6}{y^{-3}} \\ &= 8x^6y^3 \end{aligned}$$

Example 1.23 *Simplify:* $2\sqrt{8x^3}+3x\sqrt{32x}-x\sqrt{18x}$.

Solution:

$$\begin{aligned} 2\sqrt{8x^3} &= 2\sqrt{4.2.x^2.x}=4x\sqrt{2x} \\ 3x\sqrt{32x} &= 3x\sqrt{16.2x}=12x\sqrt{2x} \\ x\sqrt{18x} &= x\sqrt{9.2x}=3x\sqrt{2x} \end{aligned}$$

Therefore,

$$\begin{aligned} 2\sqrt{8x^3}+3x\sqrt{32x}-x\sqrt{18x} &= 4x\sqrt{2x}+12\sqrt{2x}-3x\sqrt{2x} \\ &= \sqrt{2x}(4x+12x-3x) \\ &= 13x\sqrt{2x}. \end{aligned}$$

Exercises

1. Evaluate:

   a. $\left(\frac{1}{8}\right)^{1/3} + \left(\frac{1}{27}\right)^{-1/3}$.

   b. $\sqrt[5]{(-243)^2}.(49)^{-1/2}$.

   c. $10\sqrt{3x} - 2\sqrt{75x} + 3\sqrt{243x}$.

   d. $3\sqrt{9x^2} + 2\sqrt{16x^2} - \sqrt{25x^2}$.

2. Simplify, and express all answers with positive exponents.

   a. $\left(\frac{64x^6}{y^{-9}}\right)^{2/3}$.

   b. $\frac{a^2b^{-1/2}c^{1/3}}{a^{-4}b^{-1}c^{-2/3}}$.

   c. $\frac{2}{3}(3x-1)^{-1/3}.3$.

Fundamental Operations With Polynomials

A polynomial is an algebraic expression like

$$-3x^2 + 2x + 4, \quad 17x^8 - 3x^3 + 2x - 5, \quad 10t + 1, \quad 5$$

More formally, A polynomial in one variable is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

where n is a positive integer, x is a variable, and $a_0,\ a_1,\ ...,\ a_n$ are real numbers called the coefficients of the polynomial. The degree of a polynomial is the highest degree of any of its nonzero terms. For example $-p^2 + 3p - 3$ is a polynomial of degree 2, while $4x^4 - 3x + 1$ is a polynomial of degree 4.

Adding or subtracting polynomials involves the combining of **like terms.** This can be accomplished by first rearranging and regrouping the terms and then combining them.

Example 1.24 *Add:* $(4x^3 - 7x^2 + 1) + (5x^4 - 3x^3 + 2x^2 - 5)$

Solution:

$$\begin{aligned} & (4x^3 - 7x^2 + 1) + (5x^4 - 3x^3 + 2x^2 - 5) \\ = \; & 5x^4 + (4x^3 - 3x^3) + (-7x^2 + 2x^2) + (1 - 5) \\ = \; & 5x^4 + x^3 - 5x^2 - 4 \end{aligned}$$

Example 1.25 *Subtract:* $\left(5x^5 - 3x^2 + x - 7\right) - \left(3x^4 + 4x^3 - 8x + 2\right)$

Solution:

$$\begin{aligned} & \left(5x^5 - 3x^2 + x - 7\right) - \left(3x^4 + 4x^3 - 8x + 2\right) \\ = \; & 5x^5 - 3x^2 + x - 7 - 3x^4 - 4x^3 + 8x - 2 \\ = \; & 5x^5 - 3x^4 - 4x^3 - 3x^2 + 9x - 9 \end{aligned}$$

Example 1.26 *Multiply:* $(3x^3 - 8x + 4)$ *by* $\left(2x^4 - 3x + 2\right)$.

Solution:

$$\begin{aligned} & \left(3x^3 - 8x + 4\right)(2x^4 - 3x + 2) \\ = \; & 3x^3(2x^4) - 3x^3(3x) + 3x^3(2) - 8x(2x^4) - 8x(-3x) - 8x(2) + 4(2x^4) + 4(-3x) + 4(2) \\ = \; & 6x^7 - 9x^4 + 6x^3 - 16x^5 + 24x^2 - 16x + 8x^4 - 12x + 8 \\ = \; & 6x^7 - 16x^5 - x^4 + 6x^3 + 24x^2 - 28x + 8 \end{aligned}$$

Exercises

1. Simplify:
   a. $\left(2x^3y^2 - 5xy + x^2y^3\right) + \left(3xy - x^2y^3\right)$
   b. $\left(5x - 2xy + x^2y^2\right) - \left(2x + xy - x^2y^2\right)$
   c. $\left(\frac{2}{3}x + 6\right)\left(-x^2 + 3x - 5\right)$
   d. $\left(x^3 - 3x^2 + 5\right)\left(-x^4 + 7x^2 + 5x - 4\right)$

Factoring

The number 18 can be written as a product in several ways: 9.2, $(-3)(-6)$, 1.18, etc. The numbers in each product $(9,\ 2, -3,\ \text{etc.})$ are called factors and the process of writing 18 as a product of factors is called factoring. Thus, factoring is the reverse of multiplication.

Factoring of polynomials is also important. It provides a means to simplify many situations and to solve certain types of equations.

important properties in factoring:

1. $x^2 + 2xy + y^2 = (x + y)^2$
2. $x^2 - 2xy + y^2 = (x - y)^2$
3. $x^2 - y^2 = (x - y)(x + y)$
4. $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
5. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Example 1.27 *Factor the following:*

1. $9x^2 - 4$

   Solution: Notice that $9x^2 - 4$ is the difference of two squares, since $9x^2 = (3x)^2$ and $4 = 2^2$. Using Property (3) above we get that
   $$9x^2 - 4 = (3x - 2)(3x + 2)$$

2. $75m^2 + 100n^2$

   Solution: We notice that 25 is a common factor in this problem.
   $$75m^2 + 100n^2 = 25(3m^2 + 4n^2)$$

3. $128a^2 - 98b^2$
   $$\begin{aligned} 128a^2 - 98b^2 &= 2(64a^2 - 49b^2) \\ &= 2(8a - 7b)(8a + 7b) \end{aligned}$$

4. $k^3 - 8$
   $$k^3 - 8 = (k - 2)(k^2 + 2k + 4)$$

5. $16a^2 - 100 - 48ac + 36c^2$

$$\begin{aligned} & 16a^2 - 100 - 48ac + 36c^2 \\ = \ & 4\left[4a^2 - 25 - 12ac + 9c^2\right] \\ = \ & 4\left[(4a^2 - 12ac + 9c^2) - 25\right] \\ = \ & 4\left[(2a - 3c)^2 - 25\right] \\ & = 4(2a - 3c + 5)(2a - 3c - 5) \end{aligned}$$

Exercises

Factor out the following problems completely:

1. $12r + 9k$

2. $9x^3 - 36x^2 + 15$

3. $8x^3 - 64$

4. $2x^2 + 3x - 5$

5. $x^3 + 27$

2 Linear and Quadratic Functions with Applications

Introduction to Functions

To understand the origin of the concept function, we consider some "real life" situations in which one numerical quantity depends on another.

Example 2.1 *The amount of income tax you pay depends on the amount of your income.*

Example 2.2 *Suppose a rock is dropped straight down from a high point. We know from physics that the distance depends on the time.*

The first common feature shared by these examples is that each example involves two sets of numbers, one set we call input and the other set we call output.

The second common feature is that in each example there is a definite rule by which each input determines an output. The formal definition of function has these same common features (input/rule/output), with a slight change of terminology.

Definition 2.1 *A function consists of a set of input numbers called the* ***domain****, a set of output numbers called the* ***range****, and a rule by which each input number in the domain determines exactly one output number in the range.*

Example 2.3 *The following rules are function*

1. Enter a number in a calculator and press the x^2 key.

2. Assign to each input number x the number y given by the following table.

x	1	0	-3	2	4
y	2	3	1	5	7

Example 2.4 *The following rule is not a function:*

Assign to each input number x the number y given by the following table.

x	1	0	-3	2	1
y	2	3	1	5	7

Since the input value 1 has two outputs (2 and 7) then this is not a function.

Example 2.5 *Decide whether each of the following equations define y as a function of x. Give the domain of each function.*

1. $y = 7x + 11$

   Solution: For a given value of x, calculating $7x + 1$ produces exactly one value of input y. Therefore, $y = 7x + 11$ defines a function with domain given by the all real numbers $(-\infty, \infty)$.

2. $y^2 = x$

   Solution: Let $x = 16 = y^2$. Then $y = 4$ or $y = -4$. Since one value of x has two output values of y then this is not a function.

Definition 2.2 *The domain of any function is the largest set of real numbers that will produce a well-defined output y.*

Functional Notation: Functions are usually denoted by a letter f. If x is an input then $f(x)$ is the output number that the function f produces from the input x.

Example 2.6 *Find the domain of the following functions.*

1. $f(x) = x^4$

Solution: Since any number may be raised to the fourth power, the domain is $(-\infty, \infty)$.

2. $f(x) = \sqrt{12 - 2x}$

   Solution: For $f(x)$ to be defined, $12 - 2x$ must be nonnegative. That means
   $$\begin{aligned} 12 - 2x &\geq 0 \\ 12 &\geq 2x \\ 6 &\geq x \end{aligned}$$
   Hence the domain is $x \leq 6$. In interval notation, the domain is $(-\infty, 6]$.

3. $f(x) = \frac{3x+2}{2x-4}$

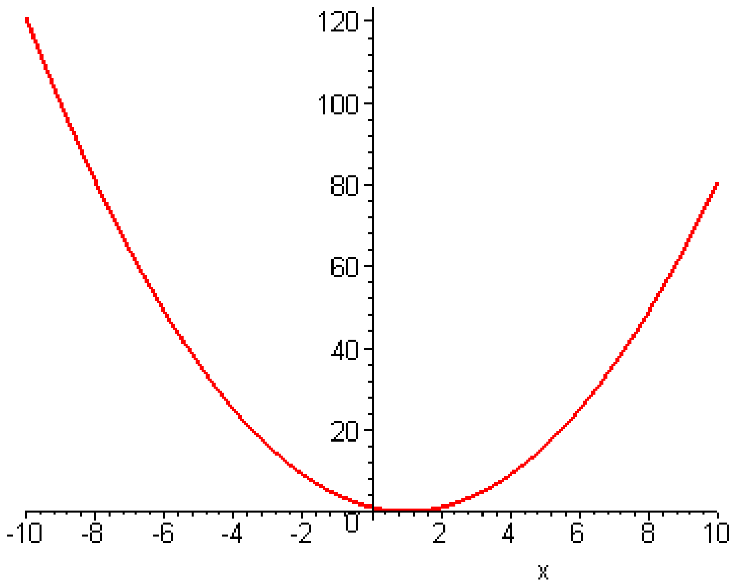
   Solution: Since the denominator cannot be zero, then $2x - 4 \neq 0 \Leftrightarrow x \neq 2$. Therefore, the domain is $(-\infty, 2) \cup (\infty, 2)$.

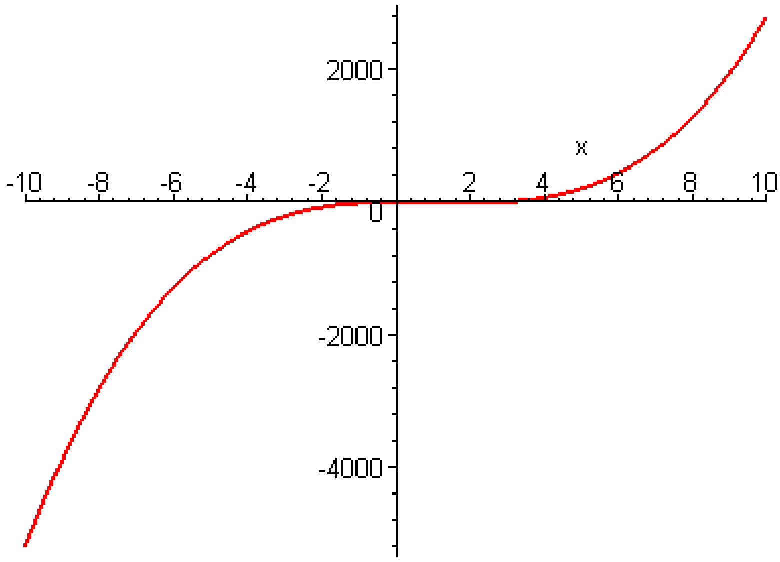
The following fact is useful to distinguish function graphs from other graphs.

Vertical Line Test

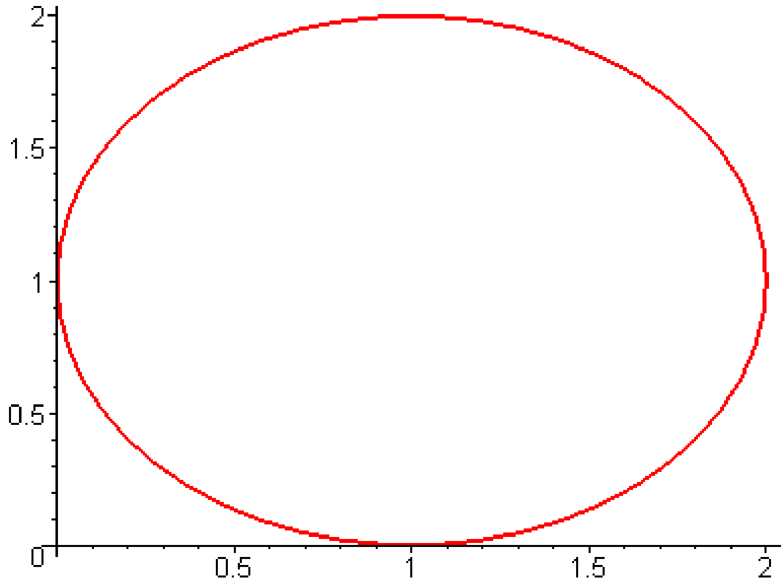
An expression f is a function if and only if any vertical line interests the graph of f at most once.

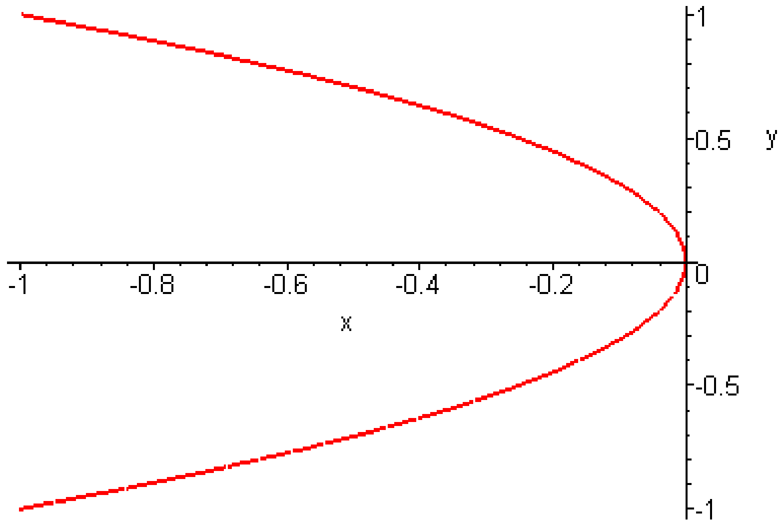
Example 2.7 *The following graphs represent functions:*





Example 2.8 *The following graphs are not functions:*





Exercises

1. State the domain of each function.

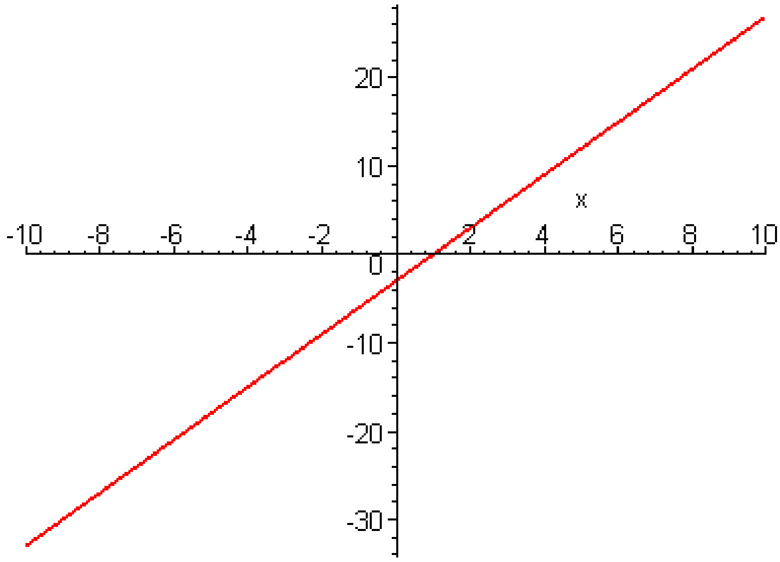
   a. $f(x) = x^4 - \sqrt{3}x + \frac{3}{\sqrt{5}}$

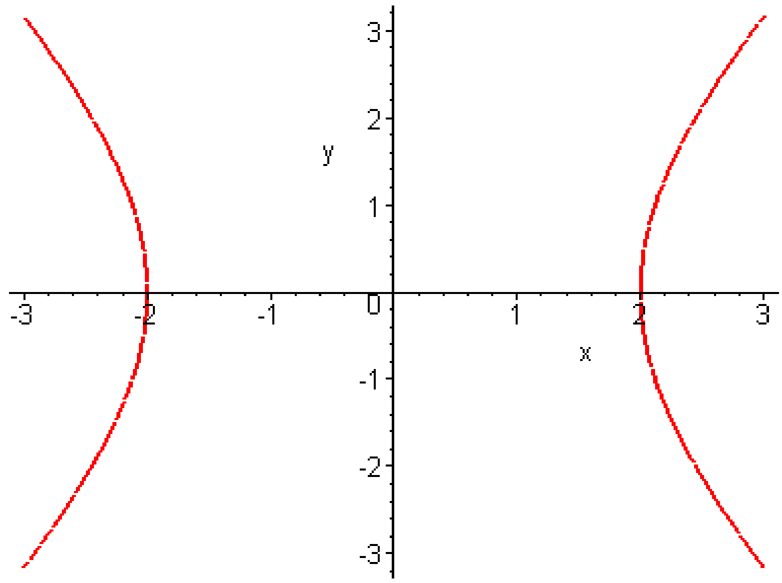
   b. $\sqrt{5 - 2x}$

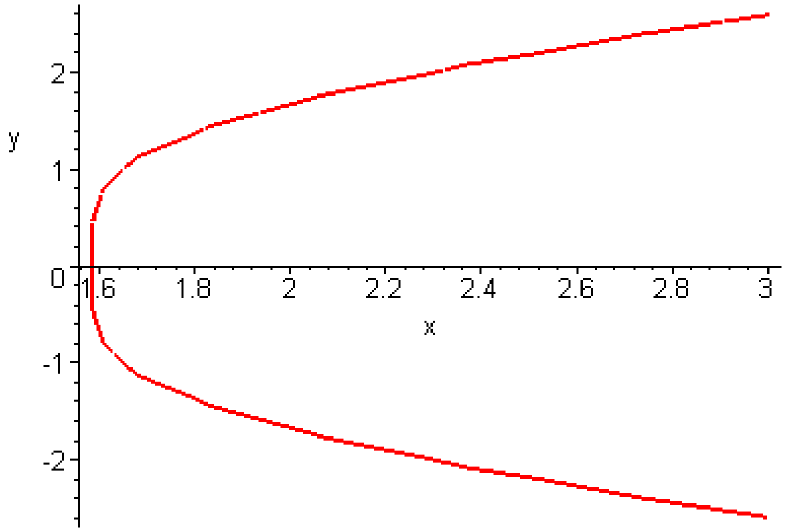
   c. $\dfrac{3x+1}{2x-7}$

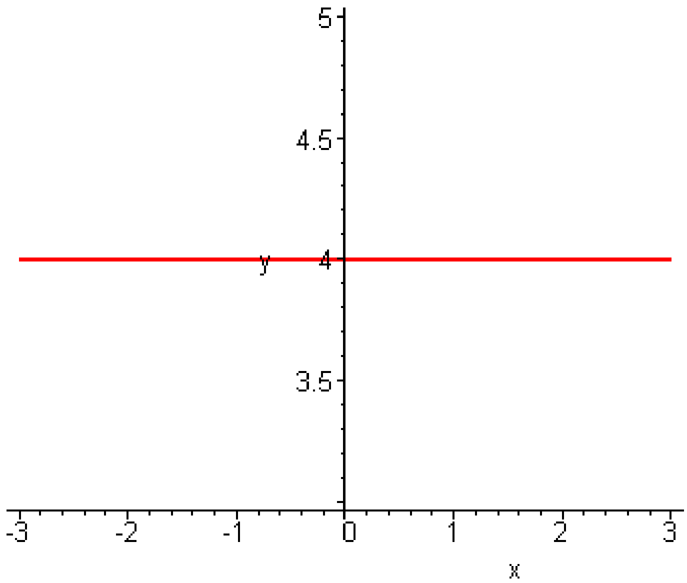
   d. $|-6 - 4x|$

2. Which of these are graphs of functions?


20
10
x
-10
-8
-6
-4
-2
0
2
4
6
8
10
-10
-20
-30


3
2
y
1
-3
-2
-1
0
1
2
3
x
-1
-2
-3


y
2
1
0
-1
-2
1.6
1.8
2
2.2
2.4
2.6
2.8
3
x


5
4.5
y
4
3.5
-3
-2
-1
0
1
2
3
x

The straight Line Equation

Equation of first degree is described by straight line in the Cartesian plane. If we have two points (x_1, y_1) and (x_2, y_2) in the plane connected by a straight line. The vertical change $(y_2 - y_1)$ is called the rise and the horizontal change $(x_2 - x_1)$ is called the run.

Example 2.9 *Graph the linear equation* $y = 2x - 1$.

Solution: To graph a linear function, find any two points on the line and then draw the line through them.

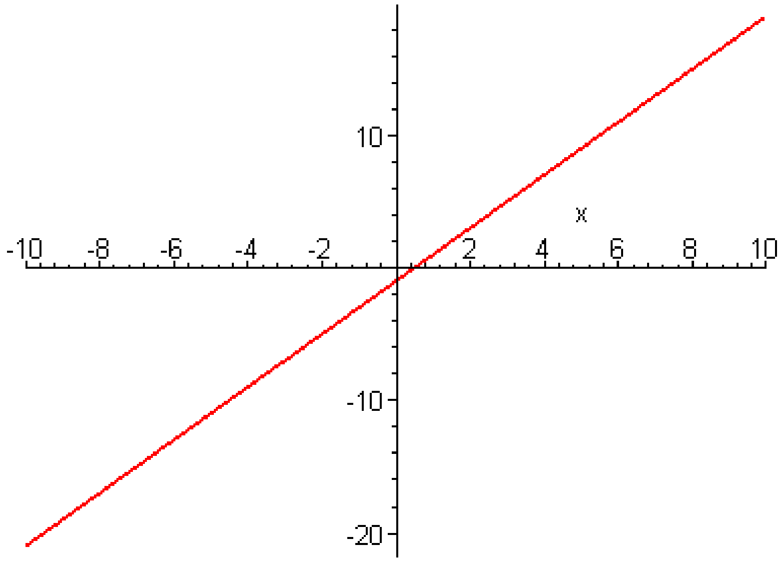
To find the $x-$intercept, let $y = 0$,

$$\begin{aligned} 2x - 1 &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

To find the $y-$intercept, let $x = 0$.

$$\begin{aligned} y &= 2(0) - 1 \\ y &= -1 \end{aligned}$$

Plot the points $(\frac{1}{2}, 0)$ and $(0, -1)$ and draw the line through them to determine the graph.



Definition 2.3 *Slope of the straight line: For* $x_1 \neq x_2$ *The* ***slope*** *of the straight line through the points* (x_1, y_1) *and* (x_2, y_2) *is the given by:*

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

Example 2.10 *let* L *be the line determined by the points* $P_1(4, 1)$ *and* $P_2(6, 9)$*. Find the slope of the line.*

solution: Substitute in the formula of the slope

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$m = \frac{9 - 1}{6 - 4} = 4.$$

Example 2.11 *Find the slope of the line who passes through the points* $(4, -5), (-2, -3)$.

Solution:

$$\begin{aligned} m &= \frac{(y_2 - y_1)}{(x_2 - x_1)} \\ m &= \frac{-3-(-5)}{-2-4} \\ m &= \frac{-3+5}{-6} \\ m &= \frac{2}{-6} \\ m &= -\frac{1}{3} \end{aligned}$$

Definition 2.4 *: Point-slope form: The equation of the line that has slope m and passes through the point (x_1,y_1) is*

$$y - y_1 = m(x - x_1)$$

Example 2.12 *Find an equation of line passing through the point $(4,-2)$ and having slope $m = 4$.*

Solution: Substituting the values $x_1 = 4$, $y_1 = -2$ and $m = 4$.

$$\begin{aligned} y-(-2) &= 4(x-4) \\ y+2 &= 4(x-4) \\ y+2 &= 4x-8 \\ y &= 4x-10. \end{aligned}$$

Example 2.13 *Find an equation of the line passing through the points $(1,-1)$ and $(3,5)$.*

Solution: First we need to find the slope of the line m.

$$\begin{aligned} m &= \frac{5-(-1)}{3-1} \\ m &= 3 \end{aligned}$$

The equation of the line is given by:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-1) &= 3(x - 1) \\ y + 1 &= 3x - 3 \\ y &= 3x - 4 \end{aligned}$$

Definition 2.5 *Slope-Intercept form: The equation of the line that has slope m and intersects the $y - axis$ at the point $(0, b)$ is :*

$$y = mx + b$$

Definition 2.6 *Horizontal Lines: The horizontal line crossing the $y - axis$ at $(0, b)$ has the equation:*

$$y = b$$

Definition 2.7 *Vertical Lines: The vertical line crossing the $x - axis$ at $(a, 0)$*

has the equation:

$$x = a$$

Example 2.14 *Find the equation of the straight line for each case .*

1. $m = 2$, and the point $P(6, 3)$ lies on the line.

   Solution:

$$\begin{aligned} y - 3 &= 2(x - 6) \\ y - 3 &= 2x - 12 \\ y &= 2x - 9 \end{aligned}$$

2. $m = -\frac{1}{2}$, and the point $(0, 3)$ lies on the line.

   solution:

$$\begin{aligned} y - 3 &= -\frac{1}{2}(x - 0) \\ y - 3 &= -\frac{1}{2}x \\ y &= -\frac{1}{2}x + 3 \end{aligned}$$

3. Horizontal line passing through $(0,-7)$.

$$y=-7$$

Exercises

1. Find the slope of the given line.

   a. $y=2x-3$

   b. $x=-2y+5$

   c. $y=12$

   d. $x=4$

2. Find the equation for the line passing through $(-3,4)$ and $(5,5)$.

3. Find the equation of the line who intersect the x-axis at $(0,-\frac{3}{4})$.

Applications of linear equations

When mathematics is used to solve real-world problems, there are no hard and fast rules that can be used in every case. You must use the given information, your common sense, and the mathematics available to you to build and appropriate mathematical model that describes the situation. In this section we focus on models given by linear equations.

Simple Interest :

Interest is the fee paid for the use of someone else's money. For example, you might pay interest to a bank for a loan you borrow. The amount of money borrowed or deposited is called the principal.

Simple interest is interest paid only on the amount deposited and not on past interest.

Definition 2.8 *The simple interest I, on an amount P dollars ate rate of interest r per year for t years is*

$$I = \mathrm{Pr}\, t$$

Definition 2.9 *The future value A of P dollars for t years all at a rate of interest r per year is:*

$$A = P(1 + rt)$$

Example 2.15 *If we borrow $6000 at the simple interest 6% per year. Find the amount owed after 5 years.*

Solution:

$$\begin{aligned} S &= P(1+rt) \\ S &= 6000(1+0.06(5)) \\ S &= 7800\$ \end{aligned}$$

Example 2.16 *Find the present value of* $\$32,000$ *in* 4 *months at* 9% *simple interest.*

Solution:

$$\begin{aligned} S &= P(1+rt) \\ 32000 &= P\left(1+0.09(\frac{4}{12})\right) \\ P &= 31067.96 \end{aligned}$$

Linear Depreciation:

When the value of a property decreases over a period of time, we call that depreciation.

If the value decreases linearly, the value of the property after t years is given by

$$\begin{aligned} V &= C(1-rt), \text{ where,} \\ V &: \text{ the future value after } t \text{ years,} \\ C &: \text{ the original cost, and} \\ r &: \text{ depreciation rate per year.} \end{aligned}$$

V: the value after t years.

C: the original cost.

r : depreciation rate per year.

Example 2.17 *An item was purchased for* 4000\$. *If the value of the item depreciates linearly at rate* 10% *per year. Find the value of the item after 6 years.*

solution:

$$\begin{aligned} V &= 4000(1-0.1(6)) \\ V &= 1600\$ \end{aligned}$$

Break -even Analysis:

In a manufacturing and sales situation the basic relationship is

Profit=Revenue−Cost

Let $R =$ The revenue received from selling x units of the product.

$C =$ The cost of manufacturing or purchasing x units of product.

then one of the following cases is going to happen:

$$\left\{ \begin{array}{ll} R > C, & \text{profit} \\ R = C, & \text{break-even} \\ R < C, & \text{loss} \end{array} \right\}$$

Example 2.18 *A manufacture of item determines that the total cost (in thousands of dollars) of producing x tons is*

$$C = 600 + 2x$$

and that the revenue (in thousand of dollars) form selling x tons is

$$R = 6x$$

1. a. Find the break -even point.
   b. At the break-even point, what are the cost and revenue?

Solution:

$$\begin{aligned} 6x &= 600 + 2x \\ 4x &= 600 \\ x &= 150 \text{ tons} \end{aligned}$$

(b) Substituting $x = 150$ in either equation yields

$C = R = 900$ (thousands of dollars)

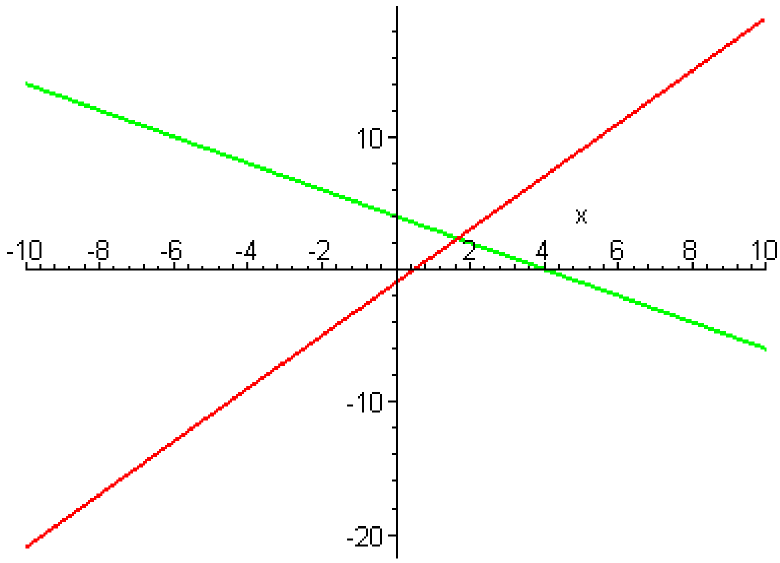
Exercises

1. A person borrows $10,000 at the simple interest rate of 6% per year.
   a. How much will be owed after 5 years?
   b. How much will be owed after 42 months?
2. A person buys a television set for 2000$ and depreciation it linearly at a certain rate per year. If the set is worth 1300$ after 4 years, what was the annual depreciation?
3. A firm producing bicycles finds that the total cost $c(x)$ of producing x bicycles is given by
   $$c(x) = 20x + 10000.$$
   Management plans to charge $24 per bicycle. How many bicycles must be sold for the firm to break even?

System of Two Linear Equations in Two unknowns:

Any two nonparallel line in the plane intersect in exactly one point. Our objective here is to find the coordinates of this point using the equations of the line.

Example 2.19 *The following graph represents a system of two linear equations in two variables.*



Two procedures will now be described to solve such systems:

The Substitution Method:

1. Solve the following system of equations by substitution:

$$\begin{aligned} 2x + 3y &= 12 \\ 4x + 2y &= 8 \end{aligned}$$

Solution: We can solve any of the equations above for x or for y and then substitute into the other equation. For example solve the second equation for y :

$$y = 4 - 2x$$

Substitute this expression into the first equation above and solve for x

$$\begin{aligned} 2x + 3y &= 12 \\ 2x + 3(4 - 2x) &= 12 \\ 2x + 12 - 6x &= 12 \\ -4x &= 0 \\ x &= 0 \end{aligned}$$

To find the $y-$value, substitute $x = 0$ into either of the original equations.

$$\begin{aligned} 2x + 3y &= 12 \\ 2(0) + 3y &= 12 \\ y &= 4 \end{aligned}$$

The solution is $(0, 4)$.

The Elimination Method:

Example 2.20 *Solve the following system by elimination:*

$$\begin{aligned} 2x - 3y &= 9 \\ 3x + 2y &= 12 \end{aligned}$$

solution: Multiply the first equation by 2 and the second equation by 3

$$\begin{aligned} 4x - 6y &= 18 \\ 9x + 6y &= 36 \end{aligned}$$

Adding the last two equations yields

$$13x = 54$$

Divided by 13

$$x = \frac{54}{13} = 4.15$$

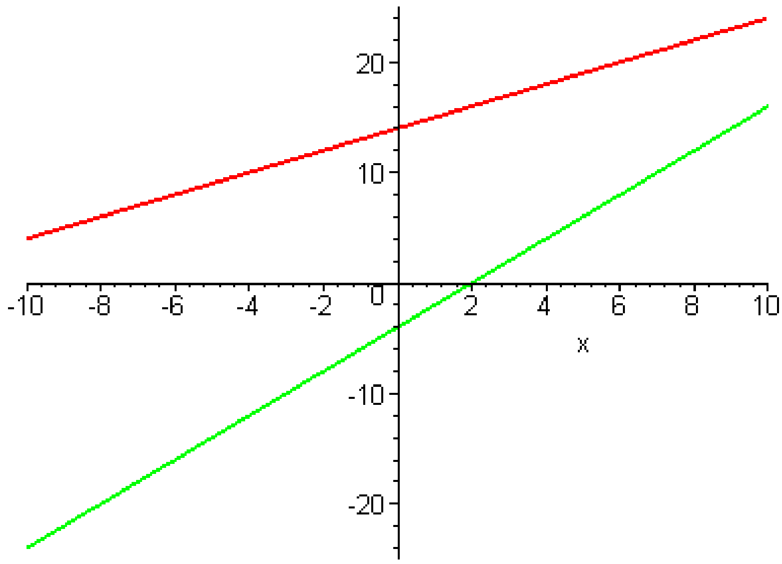
Substitute the value of x in the any equation in the original system

$$\begin{aligned} 2(4.15) - 3y &= 9 \\ -3y &= 9 - 8.3 \\ y &= -0.23 \end{aligned}$$

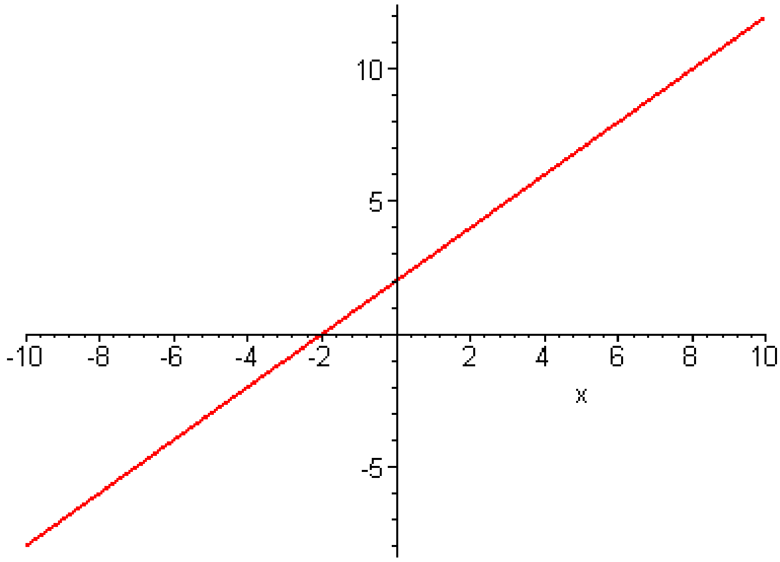
The solution is $(4.15, -0.23)$

When a linear system has a unique solution, as in the preceding example, we say that the system is consistent. Graphically, this means that the two lines intersect at exactly one point. There are two other possibilities:

1. An inconsistent system (No solution)



2. A dependent system (An infinite number of solutions)



Exercises

1. Solve the following systems of equations

   a. $2x + 3y = -5$
   $3x + 2y = 12$

   b. $5x + 3y = -5$
   $4x + 2y = -2$

   c. $-2y + 7 = 5x$
   $10x - 4y - 8 = 0$

Quadratic function

Definition 2.10 *A quadratic function is a function that can be written in the form*

$$f(x) = ax^2 + bx + c,$$

where $a,\ b,\ c$ *are constants and* $a \neq 0$.

Example 2.21 *The following functions are example of quadratic functions*

$$\begin{aligned} f(x) &= -3x^2 + 4x + 1, \\ g(x) &= x^2 + 7 \\ k(x) &= 7x^2 + 4x + 2 \\ l(x) &= -2x^2. \end{aligned}$$

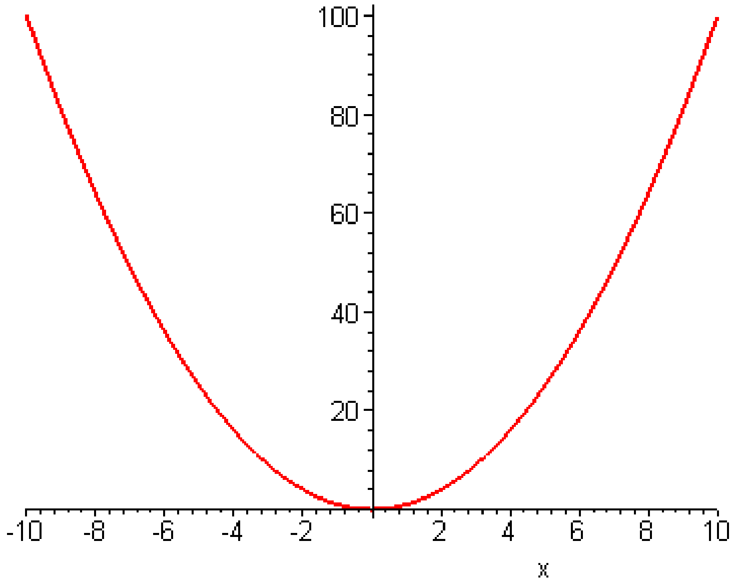
The simplest quadratic function is $f(x) = x^2$. The graph of this quadratic function will serve as the basis for drawing the graph of any quadratic function.

Example 2.22 *Graph* $f(x) = x^2$

Solution: The following table gives some points on the graph of $f(x) = x^2$

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4

When these points are located on the plane and connect them by a smooth curve, we get the following graph for $f(x) = x^2$



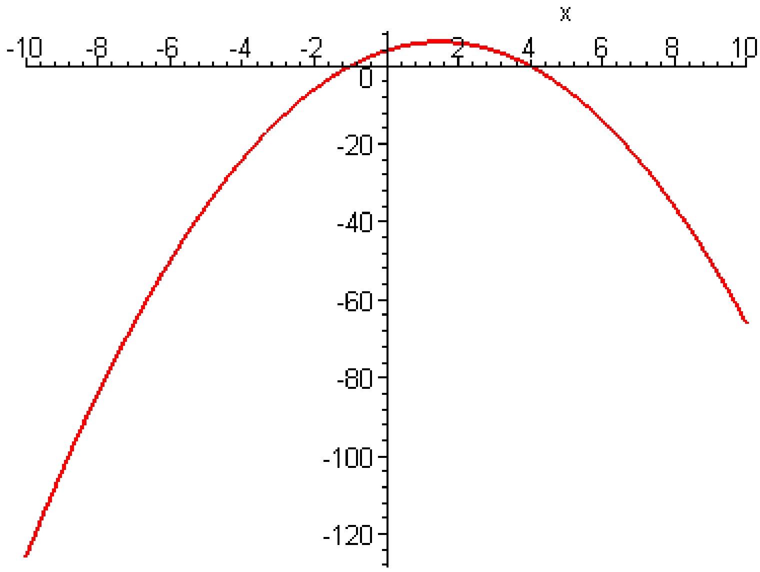
This curve is called a parabola, and every quadratic function has a parabola as its shape.

Example 2.23 *Graph* $f(x) = -x^2 + 3x + 4$

Take some value for x *and then find the corresponding values for* y

x	y
-2	-6
-1	0
0	4
1	6
2	6

Connect these points we get the following graph



From these two examples, we can conclude the following fact about quadratic functions.

- If f is a quadratic function defined by $f(x) = ax^2 + bx + x$, then the graph of $f(x)$ is a parabola having its vertex at (h, k) with $h = -\frac{b}{2a}$, and $k = f(h)$.
  a. If $a > 0$, the parabola opens upward.
  b. If $a < 0$, the parabola open downward.

Exercises

1. Sketch the graph of $f(x) = -2x^2 + 3x + 5$.

2. Sketch the graph of $g(x) = x^2 + 4$.

Quadratic Equations

An equations that can be put in the form

$$ax^2 + bx + x = 0,$$

where $a,\ b$, and c are real numbers with $a \neq 0$, is called a quadratic equation. for example, each of

$$\begin{aligned} 2x^2+3x-4 &= 0 \\ -x^2+7x-6 &= 0 \\ 3x^2+4x &= 7 \\ x^2 &= -2 \end{aligned}$$

is a quadratic equation. A solution of an equation that is a real number is said to be a real solution of the equation.

Solving Quadratic Equation

a. If $x^2=a$ and $a \geq 0$, then $x=\sqrt{a}$ or $x=-\sqrt{a}$.

b. $a^2-b^2=(a+b)(a-b)$.

Example 2.24 *Solve* $4x^2+13=253$.

Solution:

$$\begin{aligned} 4x^2+13 &= 253 && subtrarct\ 13\ from\ each\ side \\ 4x^2 &= 240 && divide\ each\ side\ by\ 4 \\ x^2 &= 60 && take\ the\ square\ root\ of\ each\ side \\ x &= \pm\sqrt{60} \end{aligned}$$

Example 2.25 *Solve* $9(x-2)^2=121$.

Solution:

$$\begin{aligned} 9(x-2)^2 &= 121 \\ (x-2)^2 &= \frac{121}{9} && Divide\ each\ side\ by\ 9 \\ (x-2) &= \pm\sqrt{\frac{121}{9}} && Take\ the\ square\ root\ of\ each\ side \\ x &= 2\pm\sqrt{\frac{121}{9}} \\ x &= \frac{13}{3}\ \ or\ \ x=-\frac{9}{3} \end{aligned}$$

Quadratic formula:

Definition 2.11 *If* $ax^2 + bx + c$ *and* $a \neq 0$, *then the solutions,or roots are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 2.26 *Use the quadratic formula to find the roots of the following:*

1) $x^2 + 5x - 14 = 0$

solution:

$$\begin{aligned} In\ \ x^2 + 5x - 14 &= 0 \quad a = 1, b = 5, and\ c = -14 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-5 \pm \sqrt{5^2 - 4(1)(-14)}}{2(1)} \\ x &= \frac{-5 + \sqrt{81}}{2} or\ \ x = \frac{-5 - \sqrt{81}}{2} \\ x &= 2 \qquad or \qquad x = -7 \end{aligned}$$

2) Solve $4x^2 = 8 - 3x$

solution:

$$\begin{aligned} 4x^2 + 3x - 8 &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-3 \pm \sqrt{3^2 - 4(4)(-8)}}{2(4)} \\ x &= \frac{-3 + \sqrt{137}}{8} \quad or \quad x = \frac{-3 - \sqrt{137}}{8} \\ x &\simeq 1.1 \qquad\qquad or \qquad x \simeq -1.8 \end{aligned}$$

Example 2.27 *Solve* $9x^2 + 30x = -25.$

Solution: Rewrite the equation in standard form as $9x^2 + 30x + 25 = 0$

$$\begin{aligned} x &= \frac{-30 \pm \sqrt{(30)^2 - (4)(9)(25)}}{2(9)} \\ &= \frac{-30 \pm \sqrt{900 - 900}}{18} \\ &= \frac{-30 \pm 0}{18} = -\frac{5}{3}. \end{aligned}$$

Therefore, the given equation has only one solution.

Example 2.28 *Solve* $x^2 + x + 2 = 0$

Solution:

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1 - 4(1)(2)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \end{aligned}$$

Since $\sqrt{-7}$ is not a real number, then this equation has no real solution.

Exercises

1. Use the quadratic formula to solve:
   a. $x^2 + 9x - 2 = -16$

   b. $-2x^2 + 4x = -2$

   c. $5x^2 - 2x - 3 = 0$

   d. $2x^2 + 3x + 4 = 0$

3 Rational functions

Rational Functions

A rational function is a function whose rule is the quotient of two polynomial such as

$$f(x) = \frac{3}{1+x},$$

$$g(x) = \frac{x^2 - 2x - 4}{x^3 - 2x^2 + x}.$$

Thus a rational function is one whose rule can be written in the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$, and $Q(x)$ are polynomials with $Q(x) \neq 0$.

Example 3.1 *Evaluate* $\frac{4}{5k} - \frac{7}{5k}$.

Solution: When two rational polynomials have the same denominator, subtract by subtracting the numerators and keeping the common denominator.

$$\frac{4}{5k} - \frac{7}{5k} = \frac{4-7}{5k} = -\frac{7}{5k}$$

Example 3.2 *Evaluate* $\frac{7}{x} - \frac{3}{2x} + \frac{4}{3x}$.

Solution: These three denominators are different. First we must find a common denominator, one that can be divided by $x, 2x$, and $3x$. A common denominator here is $6x$.

Rewrite each rational function, using the common denominator $6x$

$$\begin{aligned}\frac{7}{x}-\frac{3}{2x}+\frac{4}{3x} &= \frac{42}{6x}-\frac{9}{6x}+\frac{8}{6x}\\ &= \frac{42-9+8}{6x}\\ &= \frac{41}{6x}\end{aligned}$$

Example 3.3 *Evaluate* $\frac{3}{x^2-2x-3}+\frac{5}{x^2-x-6}$.

Solution: The denominators $(x^2-2x-3)=(x-3)(x+1)$, and $(x^2-x-6)=(x-3)(x+2)$ have $(x-3)(x+1)(x+2)$ as a common denominator. Rewrite each rational expression using the common denominator $(x-3)(x+1)(x+2)$

$$\begin{aligned}&\frac{3}{x^2-2x-3}+\frac{5}{x^2-x-6}\\ =&\ \frac{3}{(x-3)(x+1)}+\frac{5}{(x-3)(x+2)}\\ =&\ \frac{3(x+2)}{(x-3)(x+1)(x+2)}+\frac{5(x+1)}{(x-3)(x+1)(x+2)}\\ =&\ \frac{3(x+2)+5(x+1)}{(x-3)(x+1)(x+2)}=\frac{3x+6+5x+5}{(x-3)(x+1)(x+2)}\\ =&\ \frac{8x+11}{(x-3)(x+1)(x+2)}.\end{aligned}$$

Exercises

1. Evaluate $\frac{3}{m-1}+\frac{3m+2}{m-1}$.

2. Evaluate $\frac{8}{3(x-1)}-\frac{2}{(x-1)}+\frac{3}{4(x-1)}$.

3. Evaluate $\frac{3}{x^2-3x-10}-\frac{7x}{x^2-x-20}$.

Rational Equations

Example 3.4 *Solve* $\dfrac{3x}{x+7} - \dfrac{8}{5} = 0.$

Solution: Multiply both sides of the equation by the least common denominator $5(x+7)$

$$\begin{aligned} 5(x+7)\left(\frac{3x}{x+7} - \frac{8}{5}\right) &= 5(x+7)0 \\ 5(3x) - (x+7)8 &= 0 \\ 15x - 8x - 56 &= 0 \\ 7x &= 56 \\ x &= 8 \end{aligned}$$

To check if $x = 8$ is a solution, substitute this value in the original equation.

$$\frac{3(8)}{8+7} - \frac{8}{5} = \frac{24}{15} - \frac{8}{5} = \frac{24-24}{15} = 0.$$

Therefore, $x = 8$ is a solution.

Example 3.5 *Solve* $\dfrac{2x-5}{x+1} - \dfrac{3}{x^2+x} = 0.$

Solution:

$$\begin{aligned} \frac{2x-5}{x+1} - \frac{3}{x^2+x} &= 0 \\ \frac{2x-5}{x+1} - \frac{3}{x(x+1)} &= 0 \\ x(x+1)\left(\frac{2x-5}{x+1} - \frac{3}{x(x+1)}\right) &= x(x+1)0 \\ x(2x-5) - 3 &= 0 \\ 2x^2 - 5x - 3 &= 0 \\ (2x+1)(x-3) &= 0 \\ 2x+1 &= 0 \text{ or, } x - 3 = 0 \\ x &= -\frac{1}{2} \text{ or } x = 3 \end{aligned}$$

Solve $\dfrac{2}{x+6} = \dfrac{2}{x-6}$

$$\begin{aligned}
\frac{2}{x+6} &= \frac{2}{x-6} \\
(x+6)(x-6)\frac{2}{x+6} &= (x+6)(x-6)\frac{2}{x-6} \\
(x-6)2 &= (x+6)2 \\
2x-12 &= 2x+12 \\
-12 &= 12 \text{ (Impossible)}
\end{aligned}$$

Therefore, $\frac{2}{x+6} = \frac{2}{x-6}$ has no solution.

Example 3.6 *Solve* $\dfrac{x}{x-2} - 2 = \dfrac{2}{x-2}$

Solution

$$\begin{aligned}
\frac{x}{x-2} - 2 &= \frac{2}{x-2} \\
(x-2)\left(\frac{x}{x-2} - 2\right) &= (x-2)\frac{2}{x-2} \\
x-(x-2)2 &= 2 \\
x-2x+4 &= 2 \\
-x &= 2-4 \\
-x &= -2 \\
x &= 2.
\end{aligned}$$

Substituting $x = 2$ in the original equation we get $\dfrac{2}{0} - 2 = \dfrac{2}{0}$. Since $\frac{2}{0}$ is undefined we get that $x = 2$ is not a solution and hence $\dfrac{x}{x-2} - 2 = \dfrac{2}{x-2}$ **has no solution.**

Exercises

1. Solve each equation

   a. $\dfrac{8}{3k-9} - \dfrac{5}{k-3} = 4$

   b. $\dfrac{3}{2x+3} - \dfrac{2}{x-2} = \dfrac{4}{2x+3}$

   c. $\dfrac{2}{x+3} = \dfrac{3}{x+3}$

d. $\dfrac{x}{x-6} + \dfrac{2}{x} = \dfrac{1}{x-6}$

Rational Inequalities

Inequalities involving quotients of polynomials are called rational inequalities. These inequalities can be solved in much the same way as polynomial inequalities

Example 3.7 *Solve the rational inequality* $\dfrac{5}{x+4} \geq 1$

Solution: Begin by putting all nonzero terms on the left side of the inequality and writing them as a single fraction.

$$\begin{aligned}
\frac{5}{x+4} - 1 &\geq 0 \\
\frac{5}{x+4} - \frac{x+4}{x+4} &\geq 0 \\
\frac{5-(x+4)}{x+4} &\geq 0 \\
\frac{5-x-4}{x+4} &\geq 0 \\
\frac{1-x}{x+4} &\geq 0
\end{aligned}$$

The quotient on the left can change sign only when the denominator is 0 or the numerator is 0, that is when, $x = -4$ or $x = 1$. These numbers separate the real line into three intervals, as shown in the following graph.

On each of these intervals, $\dfrac{1-x}{x+4}$ is either always positive or always negative. Determine its sign in each interval by choosing a number in each interval and evaluating it there

$$\begin{aligned}
\text{Let } x &= -5 \text{ in } (-\infty, -4): & \frac{1-(-5)}{-5+4} &= -6 < 0 \\
\text{Let } x &= 0 \text{ in } (-4, 1): & \frac{1-0}{0+4} &= \frac{1}{4} > 0 \\
\text{Let } x &= 2 \text{ in } (1, \infty): & \frac{1-2}{2+4} &= \frac{-1}{6} < 0
\end{aligned}$$

Therefore, the solution for $\frac{5}{x+4} - 1 \geq 0$ is $(-4, 1]$.

1.

Example 1: Solve $\dfrac{4}{x+1} > -1$

$$\begin{aligned}
\frac{4}{x+1} + 1 &> 0 \\
\frac{4+x+1}{x+1} &> 0 \quad \text{Get a common denominator} \\
\frac{5+x}{x+1} &> 0 \quad \text{Study the sign of each term} \\
5+x &> 0 \text{ when } x > 5 \\
x+1 &> 0 \text{ when } x > 1 \\
\text{Therefore, } \frac{5+x}{x+1} &> 0 \text{ when } x \in (-\infty, 1) \cup (5, \infty)
\end{aligned}$$

Example 2: Solve $\dfrac{x^2+x-12}{x-1} \geq 0$

$$\begin{aligned}
\frac{x^2-x-12}{x-1} &\geq 0 \\
\frac{(x-4)(x+3)}{x-1} &\geq 0
\end{aligned}$$

We have to study $\dfrac{x^2+x-12}{x-1}$ on each of the following intervals:

$$\begin{aligned}
\text{Let } x &= -4 \text{ in } (-\infty,-3): \quad \frac{(-4-4)(-4+3)}{-4-1} = -\frac{8}{5} < 0 \\
\text{Let } x &= 0 \text{ in } (-3,1): \quad \frac{(0-4)(0+3)}{0-1} = 12 > 0 \\
\text{Let } x &= 2 \text{ in } (1,4): \quad \frac{(2-4)(2+3)}{2-1} = -\frac{10}{1} < 0 \\
\text{Let } x &= 5 \text{ in } (4,\infty): \quad \frac{(5-4)(5+3)}{5-1} = \frac{8}{4} > 0.
\end{aligned}$$

Therefore the solution for $\dfrac{x^2+x-12}{x-1} \geq 0$ is $[-3,1) \cup [4,\infty)$.

Exercises

1. Solve each inequality
   a. $\dfrac{x+5}{x-3} < -4$

   b. $\dfrac{x^2-9}{x+2} < 0$

   c. $\dfrac{x+5}{x^2+x-42} > 0$

4 Review Examples

1. If $4^{x+y} = 8$, what is the value of $2x + 2y$?

   Solution:

$$\begin{aligned} 4^{x+y} &= 8 \\ 2^{2(x+y)} &= 2^3 \\ 2^{2x+2y} &= 2^3 \\ 2x + 2y &= 3 \end{aligned}$$

2. If $3^{x-1} + 3^{x-1} + 3^{x-1} = (81)^y$, what is the value of $\frac{x}{y}$?

   Solution:

$$\begin{aligned} 3 \cdot 3^{x-1} &= 3^{3y} \\ 3^{1+x-1} &= 3^{2y} \\ 3^x &= 3^{2y} \\ x &= 2y \\ \frac{x}{y} &= 2 \end{aligned}$$

3. Factor $6x^2 + 11x + 3$.

   Solution:

$$6x^2 + 11x + 3 = (3x + 1)(2x + 3)$$

4. Factor $3x^2 + 11x - 20$.

   Solution:

$$3x^2 + 11x - 20 = (3x - 4)(x + 5)$$

5. If $(px + q)^2 = 9x^2 + kx + 16$, what is the value of k?

Solution:

$$\begin{aligned}(px+q)^2 &= 9x^2+kx+16\\ p^2x^2+2pqx+q^2 &= 9x^2+kx+16\\ p^2 &= 9 \Rightarrow p=\pm 3,\\ q^2 &= 16 \Rightarrow q=\pm 4, \text{ and}\\ k &= 2pq \Rightarrow k=\pm 24\end{aligned}$$

6. State wether or not the following data represent a function

a.

x	y
2	-3.6
3	4.2
4	4.2
5	10.7
6	12.1

Solution: For each value of x, there is exactly one value of y. The data represents a function.

b.

x	y
3	**7**
3	**8**
4	6
3	**5**
6	5
4	5
8	2

Solution: The set of data does not represent a function, because three different y- values are paired with the same value x.

7. If $2b=-3$, what is the value of $1-4b$?

Solution:

$$\begin{aligned}2b &= -3 \qquad \text{divided by 2}\\ b &= -\frac{3}{2} \Rightarrow 1-4b = 1-4(-\frac{3}{2})\\ &= 1+6=7\end{aligned}$$

8. If $(y-3)^2=16$, what is the smallest possible value of y^2?

Solution:

$$\begin{aligned}(y-3)^2 &= 16\\ (y-3) &= \pm\sqrt{16} \quad \text{take the radical of both sides}\\ (y-3) &= \pm 4\\ y &= 7 \quad \text{or} \quad y=-1\\ y^2 &= 49 \quad \text{or} \quad y^2 = 1.\end{aligned}$$

Therefore, the smallest value of y^2 is 1.

9. If $p^2 = 16$ and $q^2 = 36$, what is the largest possible value of $p - q$?

   Solution:

$$\begin{aligned}p^2 &= 16, \text{ and } q^2 = 36\\ &\Rightarrow p = \pm 4 \quad \text{and} \quad q = \pm 6\\ &\Rightarrow \text{the largest value is } 4-(-6) = 10\end{aligned}$$

10. If eight pencils cost \$0.42, how many pencils can be purchased with \$2.10 ?

    Solution:

$$\begin{aligned}2.10 \div 0.42 &= 5\\ 8 \times 5 &= 40 \text{ pencils}\end{aligned}$$

11. If $25(\frac{y}{x}) = 4$, what is the value of $100(\frac{x}{y})$?

    Solution:

$$\begin{aligned}25\frac{y}{x} &= 4\\ \frac{y}{x} &= \frac{4}{25}\\ \frac{x}{y} &= \frac{25}{4}\\ 100\frac{x}{y} &= 100\frac{25}{4} = 625\end{aligned}$$

12. Solve $\dfrac{2x+8}{3x-2} = \frac{5}{4}$

    Solution:

$$\begin{aligned}\frac{2x+8}{3x-2} &= \frac{5}{4}\\ (2x+8)4 &= 5(3x-2)\\ 8x+32 &= 15x-10\\ -7x &= -42\\ x &= 6\end{aligned}$$

13. A number a is increased by 20% of a results in a number b .When b is decreased by $33\frac{1}{3}\%$ of b, the result is c .The number c is what percent of a?

Solution:

$$\begin{aligned} a+(20\% \;\; of \; a) &= b \\ a+0.2a &= b \\ 1.2a &= b \\ c &= b-\frac{1}{3}b=\frac{2}{3}b \Rightarrow \\ c &= \frac{2}{3}b=\frac{2}{3}(1.2a)=0.8a. \end{aligned}$$

Hence, c is 80% of a

14. Simplify $-8x^2+5(4x^2-6)$

Solution:

$$\begin{aligned} &-8x^2+20x^2-30 \\ &12x^2-30 \end{aligned}$$

15. A store offers a 4% discount if a consumer pays cash rather than paying by a credit card. If the cash price of an item is $\$84.00$, what is the credit-card purchase price of the same item?

Solution: Let x : credit card price of the item, then the cash price is equal to $0.96x$. Hence,

$$\begin{aligned} 0.96x &= 84 \\ x &= \frac{84}{0.96} \\ x &= 87.5. \end{aligned}$$

Therefore, the credit-card price of the same item is $\$87.5$.

16. After a number is increased by $\frac{1}{3}$ of its value, the result is 24. What was the original number?

Solution: Let x be the original number. Then,

$$\begin{aligned} x+\frac{1}{3}x &= 24 \\ \frac{4}{3}x &= 24 \\ x &= \frac{24(3)}{4} \\ x &= 18. \end{aligned}$$

17. $f(x)=-2(x+2)^2-3$. Evaluate

a. $f(-3)$
b. $f(1+h)$

Solution:

a.

$$\begin{aligned} f(-3) &= -2(-3+2)^2 - 3 \\ &= -2 - 3 = -5 \end{aligned}$$

b.

$$\begin{aligned} f(1+h) &= -2(1+h+2)^2 - 2 \\ &= -2(h+3)^2 - 2 \\ &= -2(h^2+6h+9) - 2 \\ &= -2h^2 - 12h - 18 - 2 \\ &= -2h^2 - 12h - 20 \end{aligned}$$

18. Find the domain of $f(x) = \sqrt{5-2x}$

Solution:

$$\begin{aligned} 5 - 2x &\geq 0 \\ 5 &\geq 2x \\ \frac{5}{2} &\geq x. \end{aligned}$$

Hence, the domain is $(-\infty, \frac{5}{2}]$.

19. Simplify $\left(\frac{q^{-1}rs^{-2}}{r^{-5}sq^{-8}}\right)^{-1}$

20. Solution:

$$\begin{aligned} &\left(\frac{q^{-1}rs^{-2}}{r^{-5}sq^{-8}}\right)^{-1} \\ =\ &\frac{qr^{-1}s^2}{r^5s^{-1}q^8} \\ =\ &\frac{s^3}{r^6q^7}. \end{aligned}$$

21. Simplify $\left(\frac{a^2b^{-3}}{x^{-1}y^2}\right)^3\left(\frac{x^{-2}b^{-1}}{a^{3/2}y^{1/3}}\right)$

Solution:

$$\begin{aligned} &\left(\frac{a^2b^{-3}}{x^{-1}y^2}\right)^3\left(\frac{x^{-2}b^{-1}}{a^{3/2}y^{1/3}}\right) \\ =\ &\left(\frac{a^6b^{-9}}{x^{-3}y^6}\right)\left(\frac{x^{-2}b^{-1}}{a^{3/2}y^{1/3}}\right) \\ =\ &\frac{a^{9/2}b^{-10}}{x^{-1}y^{19/3}} \\ =\ &\frac{a^{9/2}x}{b^{10}y^{19/3}}. \end{aligned}$$

22. Simplify $(x^2-1)^3$

Solution:

$$(x^2-1)^3 = x^6 - 3x^4 + 3x^2 - 1.$$

23. Find the solution set for $3x + 7 \leq 2x - 3$.

Solution:

$$\begin{aligned} 3x+7 &\leq 2x-3 \Leftrightarrow \\ 3x-2x &\leq -3-7 \Leftrightarrow \\ x &\leq -10. \end{aligned}$$

Therefore, the solution set is $(-\infty, -10]$.

24. Find the solution set for $|5-2x| < 4$.

Solution:

$$\begin{aligned} |5-3x| &< 4 \Leftrightarrow \\ &\Leftrightarrow -4 < 5-3x < 4 \\ &\Leftrightarrow -9 < -3x < -1 \\ 3 &> x > \frac{1}{3} \Leftrightarrow \\ \frac{1}{3} &< x < 3. \end{aligned}$$

Therefore, the solution set is $\left(\frac{1}{3}, 3\right)$.