, MTH 320, SPRING 2009

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QUESTION 1. Let f be a group-isomorphism from $(M_1, *)$ into $(M_2, \Box)$.

a) We proved in the class that if $a \in M_1$ and $\mid a \mid= n < \infty$, then $\mid f(a) \mid= n$. Now assume that $\mid a \mid= \infty$. Prove that $\mid f(a) \mid= \infty$.

b) Let H be a normal subgroup of M_1. We know that $f(H)$ is a subgroup of M_2. ONLY PROVE THAT $K = f(H)$ is normal in M_2. (hint: Let $c \in M_2$. Show that $cK = Kc$)

c) Let $b \in M_1$. Suppose that the equation $x^m = b$ has exactly k distinct solutions in M_1, say $a_1, a_2, ..., a_k \in M_1$. Prove that the equation $x^m = f(b)$ has exactly k distinct solutions in M_2. [Hint: Show that $f(a_1), f(a_2), ..., f(a_k)$ are distinct solutions to the equation. Then you must show there are no other solutions to the equation in M_2 , so take $d \in M_2$ such that $d^m = f(b)$ and show that d must be one of the $f(a_i)$.]

QUESTION 2. Show that $(U(Z_9), \times_9)$ is group-isomorphic to $(Z_3, +_3) \oplus (Z_2, +_2)$. Assume f is a group-isomorphism from $(U(Z_9), \times_9)$ into $(Z_3, +_3) \oplus (Z_2, +_2)$. What are the possibilities for $f(2)$?

QUESTION 3. I claim that $(U(Z_8), \times_8)$ is group-isomorphic to $(U(Z_{12}, \times_{12})$. Prove my claim by finding a map f from $(U(Z_8), \times_8)$ into $(U(Z_{12}, \times_{12})$ such that f is 1-1, onto, and group-homomorphism. If you do not believe me, then tell me why?

QUESTION 4. Suppose $(M, *)$ is a finite abelian group with m elements and let n be a positive integer such that $gcd(m, n) = 1$. Let f be a map from $(M, *)$ into itself (i.e., into $(M, *)$ such that $f(a) = a^n$ for every $a \in M$. Show that f is a group-isomorphism.

QUESTION 5. It is easy to prove that the intersection of two subgroups of M is a subgroup of M (so do not prove that). However

a) Let $a \in (M, *)$ such that $\mid a \mid= 28$. Hence $((a), *)$ is a cyclic group with 28 elements. Thus we know that $H = (a^{10}) \cap (a^{21})$ is a subgroup of (a). Since (a) is cyclic, we know that H is cyclic. How many elements are in H? Write down the elements in H. What are the elements in H that generate H? Now we know that $((a)/H, \wedge)$ is a group. Find all elements of $(a)/H$. CALCULATE the order of each element in $(a)/H$. Is $((a)/H, \wedge)$ group-isomorphic to $(Z_n, +_n)$ for some n? if yes, find n and tell me WHY??

We know that every subgroup of (Z, +) is of the form nZ for some $n \geq 0$. Let $H_1 = 21Z$, $H_2 = 15Z$. Hence we know that $H_1 \cap H_2$ is a subgroup of Z and thus $H_1 \cap H_2 = mZ$ for some m. Find m.

QUESTION 6. Let H be a subgroup of Z. Say $H = 10Z$. Show there is a group-homomorphism f from $(Q^*, \times)$ into $(H, +)$ such that f is onto, i.e., $Image(f) = H$. (nice..right?)

QUESTION 7. (**the NICEST of all**) Show that there is exactly one group-homomorphism, say f, from $(Z, +)$ into $(Q^*, \times)$ such that $Ker(f) \neq \{0\}$.

QUESTION 8. Calculate the order of each element in $((U(Z_{12}), \times_{12}) \oplus (Z_2, +_2))$

QUESTION 9. We know that $(Z, +)$ is a normal subgroup of $(Q, +)$. Show that $(Q/Z, \wedge)$ is an infinite group but each element in Q/Z is of finite order.

QUESTION 10. Let F be a group-homomorphism from an abelian group $(M_1, *)$ with 30 elements into $(Z_6, +_6)$. Assume that F is surjective (ONTO), i.e., $Image(F) = Z_6$. Prove that M_1 is cyclic.

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