

EXAM one, MTH 320, SPRING 2009

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QUESTION 1. Write down T or F (Do not justify your answer)

(i) If $(M, *)$ is an abelian group and $a, b \in M$ such that $| a |= 6$ and $| b |= 8$, then $| a * b |= 48$ F

(ii) $(Z_4, +_4) \oplus (Z_5^*, \times_5)$ is a cyclic group. F

(iii) Every subgroup of an abelian group is normal. T

(iv) A_6 does not contain an element of order 8. T

(v) If M is a group with n elements and k divides n, then M has a subgroup H such that H has k elements. F

(vi) If M is a cyclic group with 18 elements and k is the number of all subgroups of M with 6 elements, then $k = \Phi(6)$. F

(vii) If M is a cyclic group with 36 elements, then there are exactly 12 elements in M such that each is of order 36. T

QUESTION 2. Let $(M, *)$ be a group. Suppose that $\{e\}$ and M are the only subgroups of M. Prove that M is a cyclic group with prime number of elements.

Proof. Suppose if we have decided to assume that $\{e\}$ and M are the only subgroups of M. Since an infinite group must have infinitely many subgroups, we conclude that M is a finite group. Let $n =| M |_s$ and let $a \in M \setminus \{e\}$. Since $\{e\}$ and M are the only subgroups of M, we conclude that $(a) = M$. Hence M is cyclic. Suppose that $| M |_s = n = kl$ for some integers k, l such that $k \neq 1$ and $l \neq 1$. Then we know that M must have a subgroup of order K and it must have a subgroup of order l, which is impossible by the hypothesis. Hence n must be a prime number.

QUESTION 3. a) Let $(H, *)$ be a subgroup of $(M, *)$. Show that the identity element of H is the same as the identity element of M.

Proof. Let e_m be the identity of M and let e_h be the identity of H. We show that $e_m = e_h$. Let $a \in H$. Since $a \in H$, we know that $a * e_h = a$. Since $a \in M$, we know that $a * e_m = a$. Hence $a * e_h = a * e_m$ (in M). Since M is left-cancelative, we conclude that $e_h = e_m$.

b) Give me an example of two sets H, M such that $H \subset M$, $(M, *)$ is a monoid, $(H, *)$ is a group but the identity element of H is not the identity element of M.

Let $M = (Z_{14}, \times_1 4)$. Then M is a monoid with 1 as the identity (note M is not a group). Let $H = \{2, 4, 6, 8, 10, 12\}$. Then $H \subset M$ and we know that $(H, \times_{14})$ is a agroup with 8 as the identity.

QUESTION 4. a) Given $(M_1, *)$, $(M_2, \Box)$ are two groups. Let $a \in M_1$ such that $| a |= n$, $b \in M_2$ such that $| b |= k$. We know that $(a, b) \in (M_1, *) \oplus (M_2, \Box)$. Show that $| (a, b) |= LCM[n, k]$

Proof. Let $l = LCM[n, k]$. Since $n | l$ and $k | l$, we conclude that $(a, b)^l = (a^l, b^l) = (e_{m_1}, e_{m_2})$. Now let $m =| (a, b) |$. Thus $(a, b)^m = (a^m, b^m) = (e_{m_1}, e_{m_2})$. Since $a^m = e_{m_1}$, we conclude $n | m$. Since $b^m = e_{m_2}$, we conclude that $k | m$. Since $(a, b)^l = (a, b)^m = (e_{m_1}, e_{m_2})$ and l is the least positive integer such that $n | l$ and $k | l$, we conclude that $l = m = LCM[n, k]$.

b) Given $(M_1, *)$, $(M_2, \square)$ are two **CYCLIC** groups such that M_1 has 27 elements and M_2 has 16 elements. Let $M = (M_1, *) \oplus (M_2, \square)$. Does M have an element of order 24? if yes, how many elements in M have order 24?

Solution: By the THEOREM, we know that M is cyclic with $3^3 \times 2^4$ elements. Since $24 = 3 \times 2^3$ divides the order of M, we know there is an element $a = (m_1, m_2)$ of order 24. By staring and by the above result, an element $a = (m_1, m_2)$ in M has order 24 if and only if the order of m_1 in M_1 is 3 and the order of m_2 in M_2 is 8. Since M_1 is cyclic, we know there are exactly $\Phi(3)$ elements in M_1 of order 3. Also, since M_2 is cyclic, we know there are exactly $\Phi(8)$ elements in M_2 of order 8. Thus THERE ARE EXACTLY $\Phi(3) \times \Phi(8) = 2 \times 4$ elements in M of order 24. (Another argument that is much shorter: Since M is cyclic, we know there are exactly $\Phi(24)$ elements in M of order 24. Since $\Phi(24) = 8$, there are exactly $\Phi(24) = 8$ elements in M of order 24.)

QUESTION 5. Let $\alpha = (235)o(35146) \in S_6$. Is $\alpha \in A_6$? Find $\mid \alpha \mid$. SO EASY....Yes, order of α is 4.

QUESTION 6. a)Let $(H, *)$ be a normal subgroup of a group (M, *). We know that $(M/H, \wedge)$ is a group. Let $a \in M$. Then $a * H \in M/H$. Suppose $\mid a \mid = k < \infty$. Show that $\mid a * H \mid$ divides k.

Proof. Since $\mid a \mid = k$, we know that $(a * H)^k = a^k * H = e * H = H$. Since H is the identity of the group M/H and $a * H$ is an element of M/H, we know that $\mid a * H \mid$ must divide k.

b) Let $(M, *)$ be an abelian group with $q_1 \times q_2$ elements, where q_1 and q_2 are two distinct prime numbers. PROVE that M is cyclic.

PROOF. Let $a \in M$ such that $a \neq e$. If $\mid a \mid = q_1 \times q_2$, then there is nothing to prove. Hence assume that $\mid a \mid \neq q_1 \times q_2$. Thus $\mid a \mid = q_1$ or q_2. We may assume that $\mid a \mid = q_1$. Thus $H = \{a, a^2, a^3, ..., a^{q_1} = e\}$ is a subgroup of M. Since M is abelian, H is normal. Hence $(M/H, \wedge)$ is a group with exactly q_2 elements (note number of elements in M/H = number of elements in M divided by number of elements in H). Since q_2 is prime, M/H is a cyclic group. Hence there is an element say $b * H \in M/H$ such that $\mid b * H \mid = q_2$, and $M/H = (b * H)$. By (a) above, we know $q_2 = \mid b * H \mid$ must divide $\mid b \mid$. Hence either $\mid b \mid = q_1 \times q_2$ or $\mid b \mid = q_2$. If $\mid b \mid = q_1 \times q_2$, then $M = (b)$ and we are done. Hence assume $\mid b \mid = q_2$. We already know that H is cyclic and $H = (a)$ and $\mid a \mid = q_1$. Since M is abelian, $a * b = b * a$. Since $gcd(\mid a \mid, \mid b \mid) = 1$, we know that $\mid a * b \mid = q_1 \times q_2$. Thus $M = (a * b)$ is cyclic.

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