$\mathbb{R}^n$

$\mathbb{R}^2 = \{ (x, y) \mid x, y \in \mathbb{R} \}$
= set of all points in xy-plane

$\mathbb{R}$ = set of all real numbers

$\mathbb{R}^3 = \{ (x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R} \}$

= set of all points where each point consists of 3 coordinates
= set of all points in the $x_1 x_2 x_3$ - plane
OR xyz-plane

$\mathbb{R}^n = \{ (x_1, \ldots, x_n) \mid x_1, \ldots, x_2 \in \mathbb{R} \}$

$(1, 2, 3) \in \mathbb{R}^3$
belong

$(5, 10) \in \mathbb{R}^2$

$(1, 7, -2, 13) \in \mathbb{R}^4$

SCALAR
$(\mathbb{R}, +, \cdot)$

$(2, 3) + (5, 7) = (7, 10)$

$-4\ (5, 10) = (-20, -40)$

$(1, 2, 0) + (-1, 3, 4) = (0, 5, 4)$

$\{ \}$ = set $\longrightarrow$ order is not imp.

$(\mathbb{R}^n, +)$ is closed under +

$(x_1, \ldots, x_n) + (y_1, \ldots, y_n)$
$= (x_1 + y_1, \ldots, x_n + y_n) \in \mathbb{R}^n$

SCALAR VIP
$(\mathbb{R}^n, \cdot)$ is closed under $\cdot$

span $\{ \underbrace{(2, 1, 3), (0, 1, 5)}_{\in \mathbb{R}^3} \}$ = set of <u>ALL</u> linear combination of $(2, 1, 3), (0, 1, 5)$

linear combination of $(2,1,3)$, $(0,1,5)$ means
$c(2,1,3) + c(0,1,5)$

does $3(2,1,3) + (0,1,5) \in D$?

yes, $(6,3,9) + (0,1,5) = (6,4,14)$

does $\sqrt{2}\,(2,1,3) + -4\,(0,1,5) \in D$? yes

D "lives" inside $\mathbb{R}^3$ $\therefore$ D "subset" of $\mathbb{R}^3$

Def: let D be a subset of $\mathbb{R}^n$. D is called a subspace of $\mathbb{R}^n$ <u>IF</u> $D = \text{span}\{$ finite # of points in $\mathbb{R}^n \}$

$$0(1,2,1,0) = (0,0,0,0) \in D$$

is $(1,4,2\alpha) \in D$?

so, can we find a number c such that
$c(1,2,1,0) = (1,4,2,0)$?

$$(c, 2c, c, 0) = (1,4,2,0)$$

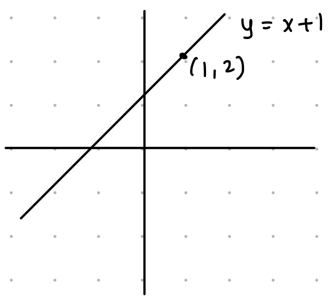
$c = 1$
$2c = 4 \therefore c = 2$ impossible so, NO !

$(1,4,2,0)$ is a point in $\mathbb{R}^4$
$\therefore$ D "subspace" of $\mathbb{R}^4$
but not $= \mathbb{R}^4$

$$D = \text{span}\{Q_1, \dots, Q_k\}$$

$Q_1, \dots, Q_k$ are points in $\mathbb{R}^n$

$(0,0,0,\dots,0) \in D$ always true !
$c = 0$

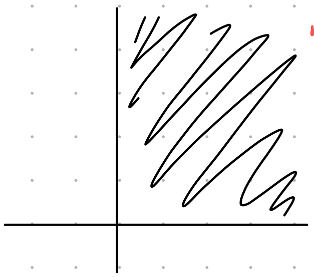


$D = \{(x, x+1) \mid x \in \mathbb{R}\}$
set of all points on the line $y = x+1$
D is a subset of $\mathbb{R}^2$ but D is NOT a subspace of $\mathbb{R}^2$ bcuz we cant span of a finite # of points

if D is a subspace by our def: D = span { some points }

$\Rightarrow$ (0, 0) $\in$ D but (0, 0) $\notin$ D

hence D cant be a subspace



all the points in the first quadrant

$D = \{ (x, y) \mid x \geq 0 \ \ y \geq 0 \}$

D is a subset of $\mathbb{R}^2$
but not a subspace

Sol: assume D is a subspace

by def. $D = \text{span} \{ (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n) \}$

$x_1, \dots, x_n$
$y_1, \dots, y_n$ ≥ 0

linear comb. $\longrightarrow$ $-1 (x_1, y_1) + 0 (x_2, y_2) + \dots + 0 (x_n, y_n) \in D$

impossible $(-x_1, -y_1) + (0,0) + \dots + (0,0) \notin D$

$D = \text{span} \{ (1,0) (0,1) \}$

D is a subspace of $\mathbb{R}^2$

anything written in the lang of span is a subspace

subspace $\longrightarrow$ subset

subset $\xrightarrow{\text{may or may not}}$ subspace

every point in $\mathbb{R}^2$ can be written as a linear combination of (0,1) (1,0)

$(\sqrt{2}, \sqrt{5}) = \sqrt{2} (1,0) + \sqrt{5} (0,1) = (\sqrt{2}, \sqrt{5})$

$(a, b) = a (1,0) + b (0,1) = (a, 0) + (b, 0) = (a, b)$

Subspaces:

Q: $D = \{ (x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R} \}$

D "lives" in $\mathbb{R}^3$

the set D is infinite

Use the concept of span and show that D is a subspace of $\mathbb{R}^3$

$$D = \{ x_1 (1,0,1) + x_2 (0,1,1) \mid x_1, x_2 \in \mathbb{R} \}$$

$$= \text{span} \{ (1,0,1), (0,1,1) \}$$

if he doesn't specify which span method then we can use whichever

$D = \{ (x_1, x_2, -2x_1 + 3x_2, x_4) \mid x_1, x_2, x_4 \in \mathbb{R} \}$

D lives inside $\mathbb{R}^4$ & D is an infinite set

convince me that D is a subspace of $\mathbb{R}^4$

$$D = \{ x_1 (1,0,-2,0) + x_2 (0,1,3,0) + x_4 (0,0,0,1) \mid x_1, x_2, x_4 \in \mathbb{R} \}$$

$$= \text{span} \{ (1,0,-2,0), (0,1,3,0), (0,0,0,1) \}$$

$D = \{ x_1, x_2, x_1 + 2 \mid x_1, x_2 \in \mathbb{R} \}$

D is not a subspace

D is a subset of $\mathbb{R}^3$

here is 1 so we cant span

$$D = \{ x_1 (1,0,1) + x_2 (0,1,0) + (0,0,2) \mid x_1, x_2 \in \mathbb{R} \}$$

$\neq$ Span

ANOTHER METHOD:

Check $(0,0,0)$ lives in D

$(0, 0, 0+2) = (0,0,2)$ so span is impossible

$D = \{ x_1, x_1 - x_3, x_3 \mid \in x_1, x_3 \in \mathbb{R} \}$

D lives in $\mathbb{R}^3$

not finite

$$D = \{ x_1 (1, x_3, 0) + x_3 (0, x_1, 1) \mid x_1, x_3 \in \mathbb{R} \}$$

$\neq$ span { finite points }

Linear transformation

($\mathbb{R}$ - homomorphism)

$$f : \underset{\text{x-axis}}{\overset{\text{domain}}{\mathbb{R}}} \longrightarrow \underset{\text{y-axis}}{\overset{\text{co-domain}}{\mathbb{R}}}$$

$f(x) = x + 3$

Q: $\overset{\text{func}}{T} : \overset{\text{domain}}{\mathbb{R}^2} \longrightarrow \overset{\text{co-domain}}{\mathbb{R}}$

$T(x_1, x_2) = 2x_1 - 5x_2$

Show T is a linear transformation

illustrate

$T(1,3) = 2(1) - 5(3) = -13$

$T(2,1) = 2(2) - 5(1) = -4$

$T(3,4) = 2(3) - 5(4) = -14$

$T(\underbrace{(1,3) + (2,1)}_{(3,4)}) = T(1,3) + T(2,1)$

$= -13 + -1 = -14$

$T : \mathbb{R} \longrightarrow \mathbb{R}$

$T(x) = x + 1$

$T(2) = 3 \qquad T(4) = 5$

$T(6) = 7 \neq T(2) + T(4)$

Def: $T : \overset{\text{domain}}{\mathbb{R}^n} \longrightarrow \overset{\text{co-domain}}{\mathbb{R}^m}$

$\mathbb{R}$ - homomorphism

is called linear transformation IF

(1) $T(Q_1 + Q_2) = T(Q_1) + T(Q_2)$

for every points Q_1, Q_2 in $\mathbb{R}^n$

constant

(2) $T(c, Q) = c\,T(Q)$

for every real # c and every point Q in $\mathbb{R}^n$

Q: $T: \mathbb{R} \to \mathbb{R}$

$T(x) = 3x$

is T a linear transformation? is T R-homomorphosim?

$a_1, a_2 \in \mathbb{R}$

check $T(a_1 + a_2) \stackrel{?}{=} T(a_1) + T(a_2)$

$T(a_1 + a_2) = 3(a_1 + a_2) = 3a_1 + 3a_2$

$T(a_1) = 3a_1$

$T(a_2) = 3a_2$

$T(a_1 + a_2) = T(a_1) + T(a_2)$

choose $c \in \mathbb{R}$, $a \in \mathbb{R}$

$T(ca) \stackrel{?}{=} cT(a)$

$3ca = 3ca$

$T: \mathbb{R} \to \mathbb{R}$

$T(x) = x^2$ not of the form mx so not L.T.

is T a linear transformation?

No, $T(1) = 1^2 = 1$

$T(2) = 2^2 = 4$

$T(1+2) \stackrel{?}{=} T(1) + T(2)$

$T(3) = 1^2 + 2^2$

$3^2 = 1 + 4$

$9 \neq 5$

note:

we cant use $T(0) = 0$ fact bcuz here it might not be a L.T.

fact:

$T: \mathbb{R} \to \mathbb{R}$ is L.T.

IF $T(x) = mx$ for some real number m.

$T: \mathbb{R} \to \mathbb{R}$, $T(x) = 3x + 2$ NOT an L.T.

sol: (1) $3x + 2$ is not of the form mx for some fixed real num.

(2) $T(1) = 3(1) + 2 = 5$

$T(-1) = 3(-1) + 2 = -1$

$T(1-1) = 3(1) + 2 + 3(-1) + 2$

$T(0) = 5 + -1$

$2 \neq 4$

fact:

$T: \mathbb{R}^n \to \mathbb{R}^m$ is an L.T.

then $\underbrace{T(0, \dots, 0)}_{n\text{-zeros}} = \underbrace{(0, 0, \dots, 0)}_{m\text{-times}}$

$T: \mathbb{R} \to \mathbb{R}$

$T(x) = 3x + 2$ continued

Sol: (3) not an L.T. since
$T(0) = 2 \neq 0$

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x_1, x_2, x_3) = (5x_1, 2x_3, x_1 + x_3)$

convince me that this is a L.T.

Sol: no sol. he didn't solve LOL

fact:

$T: \mathbb{R}^n \to \mathbb{R}^m$ is a L.T.

IF $T(x_1, \ldots, x_n) =$ (linear comb of the x_i)

i.e.: $c_1 x_1 + c_2 x_2 + c_3 x_3$

$c_1 \ldots c_3$ = some real num.

$T: \mathbb{R}^2 \to \mathbb{R}^4$

is $T(x_1, x_2) = (0, 1, x_1 + x_2, -3x_1)$ L.T. ?

Sol: 0 is a Linear Comb of x_1, x_2

$0 = 0x_1 + 0x_2$

$1 \overset{?}{=} \underbrace{c_1 x_1 + c_2 x_2}_{\text{for fixed } c_1 \text{ or } c_2}$

$x_1 + x_2 = 1 \cdot x_1 + 1 \cdot x_2$

$-3x_1 = -3x_1 + 0x_2$

No, since 1 is not a Linear combination of x_1, x_2

OR

$T(0,0) = (0, 1, 0, 0)$ so **NOT** a L.T. bcuz its $\neq (0,0,0,0)$ which is the origin

fact:

IF $T: \mathbb{R}^n \to \mathbb{R}^m$ is L.T.

then, T(origin of $\mathbb{R}^n$) = T(origin of $\mathbb{R}^m$)

$T: \mathbb{R}^3 \to \mathbb{R}$

$T(x_1, x_2, x_3) = -10x_3 + x_2$ L.T.? why?

yes, L.T.

$-10x_3 + x_2$ is a linear comb of $x_1 \dots x_3$

$-10x_3 + x_2 = 0x_1 + 1x_2 + -10x_3$

is it true $T(1,0,2) + (2,5,7) = T(1,0,2) + T(2,5,7)$

yes, its true bcuz T is a L.T.

ASK PROF TO ELABORATE & SOLVE

$T: \mathbb{R}^4 \to \mathbb{R}^5$

$T(x_1, x_2, x_3, x_4) = T(-2x_1 + 3x_2,\ x_3 - x_4,\ x_1 + 2x_2 - x_3,\ 0,\ x_4 + x_1)$ is L.T.?

yes, by staring, each coord is equal L.T. of $x_1 \cdots x_4$

$T: \mathbb{R}^3 \to \mathbb{R}^4$

$T(x_1, x_2, x_3) = (x_1 \cdot x_2, 0, x_3, x_1)$ L.T.?

Sol: NO

$x_1 x_2 \overset{?}{\neq}$ fixed $c_1 x_1$ + fixed $c_2 x_2$ + fixed $c_3 x_3$

hence T is NOT L.T.

$T(0,0,0) = (0,0,0,0)$ but thats not an L.T. bcuz if we have L.T. then this is there but the opp is not true

so this $T(0,0,0) = (0,0,0,0)$ doesn't make it an L.T.

Q (by staring):

$T: \mathbb{R}^2 \to \mathbb{R}$ is a L.T.

$T(1,1) = 5 \qquad T(-1,1) = 7$

find $T(0,2) = T((1,1) + (-1,1))$

$= T(1,1) + T(-1,1)$

$= 5 + 7 = 12$

$T(-4,4) = T(4(-1,1))$

$= 4\,T(-1,1) = 4(7) = 28$

$T(0,0) = 0$
$T(0,6) = T(3(0,2)) = 3\,T(0,2)$
$= 3(12) = 36$

$T: \mathbb{R} \to \mathbb{R}$
$T(x) = x^2 + 1$
we know T is not L.T.

Range $\longrightarrow$ $1 \leq y < \infty$
zeros of T
the range is a subset of the $\underbrace{\text{co-domain}}_{\text{y-axis}}$

x-int ? $y=0$ $x=?$
zero's of T that "live" in the domain

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$T(x_1, x_2) = (3x_2, x_1 - x_2, x_1 + 5x_2)$ by staring T is a L.T.

bcuz each coord is a linear comb of x_1 & x_2

find range of T. find zeros of T

the range "lives" in $\mathbb{R}^3$ which is the co-domain

note:
zeros of T $\approx$ $Z(T)$
$\approx$ $\ker(T)$ $\approx$ null space

Range $\{ (3x_2, x_1 - x_2, x_1 + 5x_2) \mid x_1, x_2 \in \mathbb{R} \}$

Range $= \{ x_1(0,1,1) + x_2(3,-1,5) \mid x_1, x_2 \in \mathbb{R} \}$
$= \text{span}\{ (0,1,1), (3,-1,5) \}$
↳ subspace of $\mathbb{R}^3$

is $(5,2,-1) \in$ Range of T ?

Same Q

so, can we find c_1, c_2 such that
$(5,2,-1) = c_1(0,1,1) + c_2(3,-1,5)$?

$(5,2,-1) = (0, c_1, c_1) + (3c_2, -c_2, 5c_2)$

$(5,2,-1) = (3c_2, c_1 - c_2, c_1 + 5c_2)$

the range lives inside of $\mathbb{R}^3$ but is not equal to $\mathbb{R}^3$

$3c_2 = 5 \longrightarrow c_2 = 5/3$
$c_1 - c_2 = 2 \longrightarrow c_1 - 5/3 = 2 \longrightarrow c_1 = 11/3$

$-1 \stackrel{?}{=} c_1 + 5c_2$
$= 11/3 + 25/3 = 36/3$
$-1 \neq 12$

so, the point doesn't belong in the range

fact:

range of L.T. is a subspace of the co-domain

now, zeros of T

they have to live in the domain

$$T(x_1, x_2) = \underbrace{(3x_2,\ x_1 - x_2,\ x_1 + 5x_2)}_{\text{we want this to be } 0}$$

Know:

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

zeros of $T = Z(T) = \ker(T) = \text{null of } T = \{ (x_1 \cdots x_n) \mid T(x_1 \cdots x_n) = 0 \}$

$(0, 0, \cdots, 0)$ m-times

$3x_2 = 0$

$x_2 = 0$

$x_1 - x_2 = 0$

$x_1 - 0 = 0 \quad \therefore x_1 = 0$

$Z(T) = \{ (0,0) \}$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$T(x_1, x_2, x_3) = (x_1 + 2x_3,\ x_2 - 5x_3)$

T is L.T. bcuz its a linear comb of x_1, x_2, x_3

(1) find $\ker(T) = Z(T)$

$Z(T) = \{ (x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0) \}$

$(x_1 + 2x_3,\ x_2 - 5x_3) = (0,0)$

$x_1 + 2x_3 = 0 \longrightarrow x_1 = -2x_3$

$x_2 - 5x_3 = 0 \longrightarrow x_2 = 5x_3$

$x_3 \in \mathbb{R}$

$Z(T) = \{ (-2x_3, 5x_3, x_3) \mid x_3 \in \mathbb{R} \}$

$= \{ x_3 (-2, 5, 1) \mid x_3 \in \mathbb{R} \}$

$= \text{span} \{ (-2, 5, 1) \}$

its a subspace of $\mathbb{R}^3$

fact:

Z(T) ALWAYS a subspace of the domain

(2) range of T ?

$\text{range}(T) = \{ (x_1 + 2x_3,\ x_2 - 5x_3) \mid x_1, x_2, x_3 \in \mathbb{R} \}$

$= \{ x_1(1,0) + x_2(0,1) + x_3(2,-5) \}$

$= \text{span}\{ (1,0), (0,1), \underbrace{(2,-5)} \}$ → this can be removed later on since it can be transformed as a linear comb.

range lives in $\mathbb{R}^2$ so subspace of $\mathbb{R}^3$

the co-domain

Range $= \mathbb{R}^2$

take any point (a,b) in $\mathbb{R}^2$

$(a,b) = a(1,0) + b(0,1) + 0(2,-5)$

$= (a,b)$

$D = \text{span}\{ (1,0)\ (0,1)\ (1,1) \}$

D is a subspace of $\mathbb{R}^2$

$(1,1)$ is a linear comb. of $(0,1)\ (1,0)$

$D = \text{span}\{ (1,0)\ (0,1) \}$

Def: $Q_1, Q_2, \dots, Q_K$ in $\mathbb{R}^n$

we say $Q_1, \dots, Q_K$ are independent

IF whenever $c_1 Q_1 + c_2 Q_2 + \dots + c_K Q_K = \underbrace{(0,0,0 \cdots 0)}_{n\text{-times}}$

then, $c_1 = c_2 = \dots = c_K = 0$

Def: $Q_1, \dots, Q_K$ are dependent

IF there exists on $c_i \neq 0$ such that

$c_1 Q_1 + \dots + c_i Q_i + \dots + c_K Q_K = (0, 0, \dots, 0)$

equiv Def: (practical)

$Q_1, \dots, Q_K$ in $\mathbb{R}^n$ are independent

IF none of the Q_i's is a linear combination of the remaining Q_i's

$Q_1, \ldots, Q_K$ are dependent

IF at least one of the Q_i's is a linear combination of the remaining Q_i's

- $(2,1,0)$ $(0,0,3)$ $(4,2,3)$ $\in \mathbb{R}^3$

$$(4,2,3) = 2(2,1,0) + 1(0,0,3)$$
$$= (4,2,0) + (0,0,3)$$
$$= (4,2,3)$$

$\therefore$ points are dependent

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \end{bmatrix}$$

size(A) : $\underset{\#\text{rows}}{2} \times \underset{\#\text{col}}{3}$

$$= \begin{bmatrix} 0 & 1 & 4 & 5 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

: $\underset{Q_1}{(0,1,4,5)}, \underset{Q_2}{(1,0,2,1)}, \underset{Q_3}{(0,0,1,0)}$ are indep points in $\mathbb{R}^4$

$$c_1 Q_1 + c_2 Q_2 + c_3 Q_3 = (0,0,0,0)$$

$$c_1 = c_2 = c_3 = 0$$

Row operations allowed

αR_i , $\alpha \neq 0$, multiply a row with a nonzero #.

$\alpha R_i + R_K \longrightarrow R_K$

R_i interchable with R_K

↳ the whole thingy behind this is that it will make it look diff but it wont change the sol. thus you use it to solve the Q.

Q: Are $(2,4,-2)$ $(-1,2,3)$ $(0,6,4) \in \mathbb{R}^3$ indep.

solution [method]:

$$\begin{array}{c}\begin{bmatrix} 2 & 4 & -2 \\ -1 & 2 & 3 \\ 0 & 6 & 4 \end{bmatrix}\end{array}\begin{array}{l} \times \frac{1}{2} \\ \\ \\ \end{array}$$

go row by row
1^{st} row, 1^{st} non zero # needs to be "1"

$\downarrow$ equivalent

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 0 & 6 & 4 \end{bmatrix}$$

use "1" in row one and kill all #'s exactly below the "1" usually we use row op. #2

$\downarrow$ $1R_1 + R_2 \to R_2$

$$\begin{array}{c}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 6 & 4 \end{bmatrix}\end{array}\begin{array}{l} \\ \frac{1}{4} \\ \\ \end{array}$$

go to R_2 & repeat (op # 2)

$\downarrow$ $-6R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 6 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

none of the rows are $(0,0,0)$ $\therefore$ these points are indep

Q: Are $(1,2,-1,4)$ $(-2,-3,4,6)$ $(-2,-2,6,20) \in \mathbb{R}^4$ indep ?

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ -2 & -3 & 4 & 6 \\ -2 & -2 & 6 & 20 \end{bmatrix}$$

$\downarrow$ $2R_1 + R_2 \to R_2$

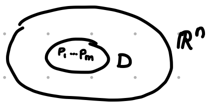
$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 2 & 14 \\ -2 & -2 & 6 & 20 \end{bmatrix} \xrightarrow{2R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 2 & 14 \\ 0 & 2 & 4 & 28 \end{bmatrix}$$

$\downarrow$ $-2R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 2 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ the points are dependent meaning that $(1,2,-1,4)$ $(-2,-3,4,6)$ is a linear comb of $(-2,-2,6,20)$

Def: let D be a subspace of $\mathbb{R}^n$, so we know $D = \text{span}\{Q_1, \dots, Q_K\}$ for some points in $\mathbb{R}^n$



$\dim(D)$ = max. # of indep. points in D (i.e.: find the independent points out of $Q_1 \cdots Q_K$)

Say $P_1, \dots, P_m$ are the max # of indep points in D.

$D = \text{span}\{P_1, \dots, P_m\}$

$\dim(D) = m$

Q: $D = \text{span}\{(1,1,0,1)\ (-2,-2,1,3)\ (0,0,1,5)\ (-2,-2,3,13)\}$

is a subspace of $\mathbb{R}^4$

(i) Find a basis for D.
(ii) Find dimension of D.
(iii) Use (i) and rewrite D

Sol:

indep
$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & -2 & 1 & 3 \\ 0 & 0 & 1 & 5 \\ -2 & -2 & 3 & 13 \end{bmatrix} \xrightarrow[2R_1 + R_4 \to R_4]{2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$
indep

$$\xrightarrow[-3R_2 + R_4 \to R_4]{-R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
indep

(ii) $\dim(D) = 2$

(i) B (basis for D) = {indep. points}
$= \{(1,1,0,1)\ (0,0,1,5)\}$

(iii) $D = \text{span}\{(1,1,0,1)\ (0,0,1,5)\}$

$30\,Q_1 + 10\,Q_2 + 15\,Q_3 - \tfrac{1}{2}\,Q_4$

(iv) is $(10,10,2,15) \in D$?

$(10,10,2,15) = c_1\,(1,1,0,1) + c_2\,(0,0,1,5)$, try to find c_1 & c_2

$= (c_1, c_1, c_2, c_1 + 5c_2)$

$c_1 = 10$

$c_1 + 5c_2 = 10 + 10 \overset{?}{=} 15$ NO such c_1, c_2 exist thus $(10,10,2,15) \notin D$

math L O L :/

1) $\mathbb{R}^n$ is a subspace of itself ($\mathbb{R}^n$, we call it vector space)

$$\mathbb{R}^n = \text{span}\{(1,0,0,0,\cdots,0)\ \underbrace{(0,1,0,\cdots,0)}_{Q_2}\ \underbrace{(0,0,1,0,\cdots,0)}_{Q_3} \cdots\cdots \underbrace{(0,0,0,\cdots,0,1)}_{Q_n}\}$$

$(a_1, a_2, \cdots, a_n)$

$= a_1Q_1 + a_2Q_2 + \cdots + a_nQ_n$

Q_2 : 2^{nd} coord $= 1$, others $= 0$
Q_3 : 3^{rd} coord $= 1$, others $= 0$
Q_n : n^{th} coord $= 1$, others $= 0$

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \cdots & & & \\ 0 & 0 & 0 \cdots & 1 \end{bmatrix}}_{n \times n} = I_n$$

$\therefore$ identity matrix

$\mathbb{R}^n = \{\text{rows of } I_n\}$

$\dim(\mathbb{R}^n) = n$

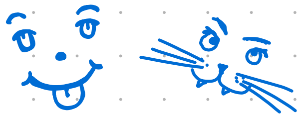
$B = \{(1,0,\cdots,0)\,(0,1,0,\cdots,0), \cdots, (0,0,\cdots,0,1)\}$
is the <u>standard basis</u> for $\mathbb{R}^n$

VIP 2) Assume D is a subspace of $\mathbb{R}^n$ and $\dim(D) = m$. Then,

(i) $\dim(D) = m \leq n$

(ii) $D = \mathbb{R}^n$ IF $n = m$

(iii) IF $k > m$, ~~any~~ then every k points in D are dependent



3) basis for $D = \{$ any m indep points in $D \}$

span $\{$ basis $\} = D$

span $\{$ any L indep points in $\mathbb{R}^n$, $L < m \} \neq D$

$D =$ span $\{$ any m indep points in $D \}$

Q: is $\{ (2,6)\ (-3,12) \}$ a basis for $\mathbb{R}^2$?

Sol:

$$\begin{bmatrix} 2 & 6 \\ -3 & 12 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 3 \\ -3 & 12 \end{bmatrix} \xrightarrow{3R_1 + R_2 \sim R_2} \begin{bmatrix} 1 & 3 \\ 0 & 21 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{21}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ yes, the points are the basis for $\mathbb{R}^2$

$\mathbb{R}^2 = \text{span} \{ (1,0)\ (0,1) \}$

$= \text{span} \{ (2,6)\ (-3,12) \}$

Questions:

Linear Transformations

subspaces

Q1: Use the concept of "span" and answer the below

(i) Is $D = \{ (x_1 + 3x_3,\ 5x_1,\ 0,\ 2x_3) \mid x_1, x_3 \in \mathbb{R} \}$ a subspace of $\mathbb{R}^4$? Is $(11, 10, 0, 6) \in D$? Is $(9, 15, 0, 8) \in D$?

$$D = \text{span} \{ x_1 (1, 5, 0, 0) + x_3 (3, 0, 0, 2) \}$$

$\therefore$ D is a subspace of $\mathbb{R}^4$

$c_1 (1,5,0,0) + c_2 (3,0,0,2)$

$(c_1, 5c_1, 0, 0) + (3c_2, 0, 0, 2c_2) \approx (11, 10, 0, 6)$

$(c_1 + 3c_2,\ 5c_1,\ 0,\ 2c_2) \approx (11, 10, 0, 6)$

$c_1 + 3c_2 = 11 \rightarrow 2 + 3c_2 = 11 \rightarrow c_2 = \frac{11-2}{3} = 3$

$5c_1 = 10 \quad \therefore c_1 = 2$

$0 = 0$

$2c_2 = 6$

$c_2 = 3$

$c_1 = 2$ & $c_2 = 3$ $\therefore$ yes it does belong to D

$(9, 15, 0, 8) \in D$?

$c_1 + 3c_2 = 9$ $\quad$ $3 + 3c_2 = 9 \rightarrow c_2 = \frac{9-3}{3} = 2$

$5c_1 = 15/5$ $\quad \therefore c_1 = 3$

$2c_2 = 8/2$ $\quad \therefore c_2 = 4$

$\therefore (9, 15, 0, 8) \notin D$ because we cant find a const c_1 & c_2

(ii) Is $D = \{ (x_1 + x_4 x_3, x_4, 0, 2x_3) \mid x_1\ x_3\ x_4 \in \mathbb{R} \}$ a subspace of $\mathbb{R}^4$?

it is not a span because of this multiplication

D is NOT a span of FINITE number of points in $\mathbb{R}^4$

(iii) is $D = \{ (x_1 + 2x_2, x_3 + 1, 0) \mid x_1, x_2, x_3 \in \mathbb{R} \}$ a subspace of $\mathbb{R}^3$?

D is NOT a subspace of $\mathbb{R}^3$ because $(0,0,0) \notin D$ therefore D is not a span of FINITE number of points.

(iv) is $D = \{ (x_1, x_3^4, x_1) \mid x_1, x_3 \in \mathbb{R} \}$ a subspace of $\mathbb{R}^4$?

NO, because of this exponent.

D is not a subspace of $\mathbb{R}^3$ because when you do span

$\{ x_1 (1, 0, 1) + x_3^4 (0, 1, 0) \mid x_1 \in \mathbb{R}$ BUT $x_3^4 \geq 0$

meaning x_3 can't be ANY real number.

(v) is $D = \{ (x_1, x_3 - 2x_4, x_1, 4x_3) \mid x_1, x_3, x_4 \in \mathbb{R} \}$ a subspace $\mathbb{R}^4$?

$D = \text{span} \{ x_1 (1, 0, 1, 0) + x_3 (0, 1, 0, 4) + x_4 (0, -2, 0, 0) \}$

$= \text{span} \{ (1, 0, 1, 0)\ (0, 1, 0, 4)\ (0, -2, 0, 0) \}$ $\quad \therefore$ yes, D is a subspace of $\mathbb{R}^4$

(vi) is $D = \{ (x_1, x_3, x_1 - 2x_3, x_4) \mid x_4 = 5x_1 - 7x_3$ & $x_1, x_3 \in \mathbb{R} \}$ a subspace of $\mathbb{R}^4$?

$D = \{ (x_1, x_3, x_1 - 2x_3, 5x_1 - 7x_3) \mid x_1, x_3 \in \mathbb{R} \}$

$= \text{span} \{ x_1 (1, 0, 1, 5) + x_3 (0, 1, -2, -7) \}$

$= \text{span} \{ (1, 0, 1, 5)\ (0, 1, -2, -7) \}$ $\quad \therefore$ yes, D is a subspace of $\mathbb{R}^4$

Q2: (i) Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(x_1, x_2, x_3) = (x_1 + 3x_2 - x_3, 2x_2 + 5)$. is L a linear transformation?

No, its not a L.T. because $2x_2+5$ is not a linear combination of x_1, x_2, x_3

&

$L(0,0,0) = (0, 5) \neq (0,0)$

(ii) Let $L: \mathbb{R}^3 \to \mathbb{R}^4$ such that
$L(x_1, x_2, x_3) = (x_1 + 3x_2 - x_3, 2x_2, \underbrace{x_1 x_3}, 0)$. is L a L.T.?

no, its not a L.T.
bcuz of this multiplication

in other words, $x_1 x_3 \neq (\text{fixed } c_1)\, x_1 + (\text{fixed } c_2)\, x_2 + (\text{fixed } c_3)\, x_3$

(iii) Let $T = \mathbb{R}^4 \to \mathbb{R}^3$ such that

$T(x_1, x_2, x_3, x_4) = (-x_2 + 4x_1 - 3x_3, 0, x_4 - 3x_1 + 2x_3)$ is T a L.T.?

write the range and z(R) as span.

yes, by staring, you can see that all the coord are a linear comb. of x_1 x_2 x_3 x_4

Range = $\{(-x_2+4x_1-3x_3, 0, x_4-3x_1+2x_3) \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$

$= \text{span}\{(x_1(4,0,-3) + x_2(-1,0,0) + x_3(-3,0,2) + x_4(0,0,1)\}$

by staring x_2 & x_4 is a linear comb of x_1 & x_3

range = span $\{(-1,0,0), (0,0,1)\}$

$Z(T) = \ker(T) = \text{Null}(T)$ is a set of all points in the domain s.t.

$T(\text{point in } \mathbb{R}^4) = (0,0,0)$

$T(x_1, x_2, x_3, x_4) = (-x_2+4x_1-3x_3, 0, x_4-3x_1+2x_3) = (0,0,0)$

$-x_2 + 4x_1 - 3x_3 = 0$
$x_2 = -(-4x_1 + 3x_3)$
$= 4x_1 - 3x_3$

$x_4 - 3x_1 + 2x_3 = 0$
$x_4 = 3x_1 - 2x_3$

$x_1, x_3 \in \mathbb{R}$

$Z(T) = \{(x_1, 4x_1 - 3x_3, x_3, 3x_1 - 2x_3) \mid x_1, x_3 \in \mathbb{R}\}$

$T: \mathbb{R} \to \mathbb{R}$

$T(x) = 3x$ is L.T. IF item 3 is eigen value

Def:

if $T: \mathbb{R}^n \to \mathbb{R}^n$ is a L.T. A number α is called an eigen value & T IFF $\exists$ a none zero point Q in the domain such that

$$T(x_1, \dots, x_n) = \alpha(x_1, \dots, x_n)$$

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x_1, x_2, x_3) = (5x_1, 3x_2, -10x_3)$

find all the eigen values

$T(1,0,0) = \boxed{?}\ (5,0,0)$

$= 5(1,0,0)$ $\therefore$ 5 = eigen value of T

3 and -10 are eigen value

any point in the span $\{(1,0,0)\}$ satisfy T(point) = 5 point

span $\{(1,0,0)\}$ eigen spaces corresponds to eigen value 5

span $\{(0,1,0)\} \Rightarrow 3$

span $\{(0,0,1)\} \Rightarrow -10$

MATRIX:

$$\begin{bmatrix} 1 & 2 & 6 & 8 \\ 0 & 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 23 & 4 & 19 \\ 8 & 1 & 6 \end{bmatrix}$$

2×4 $\quad$ 4×3

$$\overbrace{\begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix}}^{A} \quad \left.\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\} B$$

$$1 \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \times \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + 1 \times \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

∴ this a linear combination of the columns of A

using LC method

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix} = C$$

3×2

1st col:

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$$

2nd col;

$$-1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 19 & 9 \\ 8 & 8 \\ -4 & -4 \end{bmatrix}$$

∴ every column in C is a LC of columns in A

fact:

Any $N \times M$ matrices, then $T = \mathbb{R}^m \to \mathbb{R}^n$ given by

$$T(x_1, \dots, x_n) = \underset{n\times m}{M} \underset{m\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}} \longrightarrow \underset{n\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} \in \mathbb{R}^n$$

$$\therefore M \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \text{ is a L.T.}$$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$T(x_1, x_2) = [1, 4] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 + 4x_2$$

$$T(1, 3) = [1, 4] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = [1 \ 3]$$

$Z(T)$ = set

$x_1 + 4x_2 = 0$

$x_1 = -4x_2$

$$Z(T) = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 = -4x_2 , x_2 \in \mathbb{R} \}$$

$$= \{ (-4x_2 , x_2) \mid x_2 \in \mathbb{R} \}$$

$$= \{ x_2 (-4, 1) \}$$

$$= \text{Span} \{ (-4, 1) \}$$

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T(a_1, a_2, a_3) = (a_1 - 2a_2 + a_3,\ 4a_1 - 8a_2 + 4a_3)$$

(1) find the standard matrix presentation of T.

$$M = \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & -2 & 1 \\ 4 & -8 & 4 \end{bmatrix} \end{array}$$

Standard matrix presentation

Standard basis of the domain ($\mathbb{R}^3$)

$$= \{ \underbrace{(1,0,0)}_{e_1}\ \underbrace{(0,1,0)}_{e_2}\ \underbrace{(0,0,1)}_{e_3} \}$$

$T(1,0,0) = (1,4)$ 1st col of M

$T(0,1,0) = (-2,-8)$ 2nd col of M

$T(0,0,1) = (1,4)$ 3rd col of M

$$= a_1(1,4) + a_2(-2,-8) + a_3(1,4)$$

Range: span $\{(1,4)\ (-2,-8)\ (1,4)\}$

$$T(a_1, a_2, a_3) = M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Range = span {columns of A}

$$Z(T) = \{ (a_1, a_2, a_3) \in \underset{\mathbb{R}^3}{\text{domain}} \mid T(a_1, a_2, a_3) = (0,0) \}$$

$$M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a_1 - 2a_2 + a_3 = 0$$
$$4a_1 - 8a_2 + 4a_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 4 & -8 & 4 & 0 \end{array}\right] \underset{\sim}{-4R_1 + R_2 \to R_2} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

augmented matrix

completely reduced

$a_1 - 2a_2 + a_3 = 0$

$a_1 = 2a_2 - a_3$

$\underbrace{a_2, a_3}_{\text{free variables}} \in \mathbb{R}$

$Z(T) = \{ (2a_2 - a_3, a_2, a_3) \mid a_2, a_3 \in \mathbb{R} \}$

$= \{ a_2 (2,1,0) + a_3 (-1,0,1) \}$

$= \text{span} \{ (2,1,0) (-1,0,1) \}$

FACT:

$\dim(Z(T)) = \#$ of free variable when we solve $\underset{\substack{\downarrow \\ \text{standard} \\ \text{matrix presentation}}}{M} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Range $(T) = \text{span}(1,4)$

dim (Range) = 1

dim (Zeros) = 2

$\therefore \dim(\text{Range}) + \dim(\underset{Z(T)}{\text{zeros}}) = \dim(\text{Domain})$

$2 + 1 = 3 \approx \mathbb{R}^3$

$T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + x_4, 0, 2x_1 - 4x_2 + x_3 + 2x_4)$

(1) find the standard matrix presentation of T.

$$M = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 1 & 2 \end{bmatrix} \end{array}$$

each coordinate is a row

dim (codomain) x dim (domain)

$\rightarrow T(x_1, x_2, x_3, x_4) = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$T(2,4,0,1) = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -10 \end{bmatrix} = T(5,0,-10)$$

the origin of domain = $(0,0,0,0)$

" " " codomain = $(0,0,0,3)$

fact:

Rank (matrix) = # of indep rows of A
= # of indep cols of A

(2) find the rank

$-2R_1 + R_3 \to R_3$ $\sim$ $K \Rightarrow$

$$\begin{bmatrix} \boxed{1 \quad -2 \quad 0 \quad 1} \\ 0 \quad 0 \quad 0 \quad 0 \\ \boxed{0 \quad 0 \quad 1 \quad 0} \end{bmatrix}$$

→ indep → rank = 2

fact:

Row space of M = Row (M) = span { indep rows }

(3) find the rowspace :

Rowspace = Span { (1, -2, 0, 1) (0, 0, 1, 0) }

= Span { (1, -2, 0, 1) (2, -4, 1, 2) }

note:

rank (M) = dim (row (M))

(4) column space M = Col (M) = span { (1,0,0) (0,0,1) } ✗ WRONG !

this is col (K)

you must take the OG

= span { (1,0,2) (0,0,1) }

Col (M) = Range (T) = span { (1,0,2) (0,0,1) }

dim (Range (T)) = Rank (M) = dim (col (M)) = dim (row (M)) = 2

n o t e :

- T is onto IFF

  Range (T) = co-domain

- T is 1-1 IFF

  $T(Q_1) = T(Q_2)$ THEN $Q_1 = Q_2$

math :

- T is 1-1 IFF

  Z (T) = { origin of domain }
  = span { " }

- dim (span { origin }) = 0

  because its not an indep point

  ALWAYS !

T is isomorphisim :

T is onto **AND** 1-1, then T is isomorphisim

so, he is asking if its onto & 1-1.

(5) isomorphisim ? NO, because its not onto

↳ is it 1-1 ?

dim (Range) + dim (z(T)) = dim (domain)

$2 + 2 \neq 0 = 4$

∴ its not 1-1 bcuz for it to be 1-1 the dim (z(T)) = 0

$T: \mathbb{R}^4 \to \mathbb{R}^5$

$T(x_1, x_2, x_3, x_4) = (x_2 - x_3 + x_4,\ x_1 + x_2 - x_4,\ x_1 + 2x_2 - x_3,\ x_1 + x_3 + x_4,\ 0)$

(1) find all points in the domain ($\mathbb{R}^4$) s.t. T(each point) = (1,4,5,6,0)

note: it cant be onto because 5 > 4

note: T(any point) = $m \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$m = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

augmented matrix s.t. i need to find

$$m \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \\ 6 \\ 0 \end{bmatrix} \qquad [m \mid \text{constants}]$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \text{constants} \end{array} \\ \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 4 \\ 1 & 2 & -1 & 0 & 5 \\ 1 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$-R_1 + R_2 \to R_2$

$-2R_1 + R_3 \to R_3$

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 3 \\ 1 & 0 & 1 & -2 & 3 \\ 1 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$-R_2+R_3 \to R_3 \quad \sim \quad -R_2+R_4 \to R_4 \quad \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \underset{\frac{1}{3}R_4}{\sim} \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad -R_4+R_1 \to R_1 \quad \sim \quad 2R_4+R_2 \to R_2$$

$$\left[\begin{array}{cccc|c} 0 & \underline{1} & -1 & 0 & 0 \\ \underline{1} & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \text{completely reduced}$$

$\underline{x_2} - x_3 = 0$

$\underline{x_1} + x_3 = 5$

$x_4 = 1$

$0 = 0$

$\therefore x_2 = x_3$

$\therefore x_1 = 5 - x_3$

variables = 1 are leading variables $\approx x_1, x_2, x_4$

all other variables are free variables = $x_3 \in \mathbb{R}$

$\{ (5-x_3, x_3, x_3, 1) \mid x \in \mathbb{R} \}$

Questions Worksheet:

Q1: Let $L: \mathbb{R}^5 \to \mathbb{R}^4$ s.t.

$$L(x_1\ x_2\ x_3\ x_4\ x_5) = (x_2 - x_3 + x_4 + 2x_5,\ x_1 - 2x_3 - 2x_4 - 3x_5,\ -3x_1 + 6x_3 + 6x_4 + 11x_5,\ x_1 + x_2 - 3x_3 - x_4 - x_5)$$

it is clear that L is an R - homomorphism (i.e. LT)

(i) find the standard matrix representation of L.

$M = \dim(\text{codomain}) = \dim(\text{domain})$

$= 4 \times 5$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} 0 & 1 & -1 & 1 & 2 \\ 1 & 0 & -2 & -2 & -3 \\ -3 & 0 & 6 & 6 & 11 \\ 1 & 1 & -3 & -1 & -1 \end{bmatrix} \end{array}$$

(ii) rewrite L in terms of M

$$L(x_1\ x_2\ x_3\ x_4\ x_5) = M\begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 & 1 & 2 \\ 1 & 0 & -2 & -2 & -3 \\ -3 & 0 & 6 & 6 & 11 \\ 1 & 1 & -3 & -1 & -1 \end{bmatrix}$$

(iii) rewrite L in terms of M find $L(2,-1,3,-2,4)$

$$= \begin{bmatrix} 0 & 1 & -1 & 1 & 2 \\ 1 & 0 & -2 & -2 & -3 \\ -3 & 0 & 6 & 6 & 11 \\ 1 & 1 & -3 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \\ -2 \\ 4 \end{bmatrix}$$

$$= 2\begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ -2 \\ 6 \\ -3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -2 \\ 6 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -3 \\ 11 \\ -1 \end{bmatrix}$$

$$= (2, -12, 44, -10)$$

(iv) use the original def of L and find $L(2,-1,3,-2,4)$

$$L(2,-1,3,-2,4) = (\ x_2 - x_3 + x_4 + 2x_5,$$
$$x_1 - 2x_3 - 2x_4 - 3x_5,$$
$$-3x_1 + 6x_3 + 6x_4 + 11x_5,$$
$$x_1 + x_2 - 3x_3 - x_4 - x_5\ )$$

$$= (\ (-1) - 3 + (-2) + 2(4)$$
$$2 - 2(3) - 2(-2) - 3(4)$$
$$-3(2) + 6(3) + 6(-2) + 11(4)$$
$$2 + (-1) - 3(3) - (-2) - 4\ )$$

$$= (2, -12, 44, -10)$$

note: this must equal the prev (iii)

(v) is $(2,-12,44,-10) \in \text{Range}(L)$

yes, because $Q = (2,-1,3,-2,4) \in \mathbb{R}^5$

and $L(Q) = (2,-12,44,-10)$

(vi) find the rank of m

→ # of indep rows

$$\begin{bmatrix} 0 & \underset{=}{1} & -1 & 1 & 2 \\ 1 & 0 & -2 & -2 & -3 \\ -3 & 0 & 6 & 6 & 11 \\ 1 & 1 & -3 & -1 & -1 \end{bmatrix} \xrightarrow{-R_1 + R_4 \to R_4} \sim \begin{bmatrix} 0 & 1 & -1 & 1 & 2 \\ \underset{=}{1} & 0 & -2 & -2 & -3 \\ -3 & 0 & 6 & 6 & 11 \\ 1 & 0 & -2 & -2 & -3 \end{bmatrix} \begin{matrix} 3R_2 + R_3 \to R_3 \\ \sim \\ -R_2 + R_4 \to R_4 \end{matrix}$$

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 2 \\ 1 & 0 & -2 & -2 & -3 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \sim \begin{bmatrix} 0 & \underset{=}{1} & -1 & 1 & 2 \\ \underset{=}{1} & 0 & -2 & -2 & -3 \\ 0 & 0 & 0 & 0 & \underset{=}{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $(m) = 3$

(vii) find the Row (m) i.e. row space of m

span indep rows of A

$$\text{Row}(m) = \text{span}\{(0,1,-1,1,2)\ (1,0,-2,-2,-3)\ (-3,0,6,6,11)\}$$

$$= \text{span}\{(0,1,-1,1,2)\ (1,0,-2,-2,-3)\ (0,0,0,0,1)\}$$

(viii) find the col space of m: col(m)

$$\text{col}(m) = \text{span}\{(0,1,-3,1)\ (1,0,0,1)\ (2,-3,11,-1)\}$$

(ix) what is the relation b/w Rank(m), col(m), and Range(L)?
find Range(L)?

$$\dim(\text{Range}(L)) = \text{Rank}(m) = \dim(\text{col}(m)) = \dim(\text{Row}(m)) = 3$$

$$\text{Range}(L) = \text{col}(m) = \text{span}\{(0,1,-3,1)\ (1,0,0,1)\ (2,-3,11,-1)\}$$

(x) is $(4,6,0,10) \in \text{Range}(L)$?

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 6 \\ 0 & -3 & 11 & 0 \\ 1 & 1 & -1 & 10 \end{array}\right] \underset{\sim}{-R_1+R_4 \to R_4} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -6 \\ 0 & -3 & 11 & 0 \\ 0 & 1 & -3 & 6 \end{array}\right] \begin{array}{c} 3R_2+R_3 \to R_3 \\ \sim \\ -R_2+R_4 \to R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 2 & 18 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\frac{1}{2}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} 3R_3+R_2 \to R_2 \\ \sim \\ -2R_3+R_1 \to R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -14 \\ 0 & 1 & 0 & 33 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] \approx \begin{array}{l} c_1 = -14 \\ c_2 = 33 \\ c_3 = 9 \\ c_4 = 0 \end{array}$$

Valid

$0 = 0$

$\therefore$ it does $\in$ Range(L)

$$(4,6,0,10) = -14(1,0,0,1) + 33(0,1,-3,1) + 9(2,-3,11,-1)$$

(xi) find all the points in the domain ($\mathbb{R}^5$), such that

$L(\text{each point}) = (4, 6, 0, 10)$

$$L(x_1\ x_2\ x_3\ x_4\ x_5) = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -1 & 1 & 2 \\ 1 & 0 & -2 & -2 & -3 \\ -3 & 0 & 6 & 6 & 11 \\ 1 & 1 & -3 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 0 \\ 10 \end{bmatrix} \approx \text{augmented matrix}$$

$$= \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 4 \\ 1 & 0 & -2 & -2 & -3 & 6 \\ -3 & 0 & 6 & 6 & 11 & 0 \\ 1 & 1 & -3 & -1 & -1 & 10 \end{array}\right] \quad -R_1 + R_4 \to R_4 \ \sim$$

$$= \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 4 \\ 1 & 0 & -2 & -2 & -3 & 6 \\ -3 & 0 & 6 & 6 & 11 & 0 \\ 1 & 0 & -2 & -2 & -3 & 6 \end{array}\right] \quad \begin{array}{l} -R_2 + R_4 \to R_4 \\ \sim \\ +3R_2 + R_3 \to R_3 \end{array}$$

$$= \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 4 \\ 1 & 0 & -2 & -2 & -3 & 6 \\ 0 & 0 & 0 & 0 & 2 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \ \tfrac{1}{2}R_3 \quad \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 4 \\ 1 & 0 & -2 & -2 & -3 & 6 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} 3R_3 + R_2 \to R_2 \\ \sim \\ -2R_3 + R_1 \to R_1 \end{array}$$

$$= \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 0 & -14 \\ 1 & 0 & -2 & -2 & 0 & 33 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_2 - x_3 + x_4 = -14 \ \to\ x_2 = x_3 - x_4 - 14$

$x_1 - 2x_3 - 2x_4 = 33 \ \to\ x_1 = 2x_3 + 2x_4 + 33$

$x_5 = 9$ leading variables

$0 = 0$

note: NOT a subspace

$= \{(2x_3 + 2x_4 + 33,\ x_3 - x_4 - 14,\ x_3,\ x_4,\ 9) \mid x_3, x_4 \in \mathbb{R}\}$ is a set of all points in the domain $\mathbb{R}^5$ where $L(\text{points}) = (4, 6, 0, 10)$

(xii) find dim (Range (L)) and dim (Z(L))

$$\dim(\text{Range}(L)) + \dim(Z(L)) = \dim(\text{domain } L)$$

$$3 \quad + \quad ? \quad = \quad 5$$

$$\dim(Z(L)) = 2$$

(xiii) find Z(L) and write it as span of a basis

so, L (each point) = (0,0,0,0)

$$= \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 0 \\ 1 & 0 & -2 & -2 & -3 & 0 \\ -3 & 0 & 6 & 6 & 11 & 0 \\ 1 & 1 & -3 & -1 & -1 & 0 \end{array}\right] \quad -R_1 + R_4 \to R_4 \quad \sim$$

$$\left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 0 \\ 1 & 0 & -2 & -2 & -3 & 0 \\ -3 & 0 & 6 & 6 & 11 & 0 \\ 1 & 0 & -2 & -2 & -3 & 0 \end{array}\right] \quad \begin{array}{c} 3R_2 + R_3 \to R_3 \\ \sim \\ -R_2 + R_4 \to R_4 \end{array}$$

$$\left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 0 \\ 1 & 0 & -2 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{c} \frac{1}{2}R_3 \\ \sim \end{array} \quad \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 2 & 0 \\ 1 & 0 & -2 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{c} 3R_3 + R_2 \to R_2 \\ \sim \\ -2R_3 + R_1 \to R_1 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_2 - x_3 + x_4 = 0 \quad \sim \quad x_2 = x_3 - x_4$

$x_1 - 2x_3 - 2x_4 = 0 \quad \sim \quad x_1 = 2x_3 + 2x_4$

$x_5 = 0$

$0 = 0$

$$Z(L) = \{ (2x_3 + 2x_4,\ x_3 - x_4,\ x_3,\ x_4,\ 0) \mid x_3\ x_4 \in \mathbb{R} \}$$

$$= \{ x_3 (2,1,1,0,0) + x_4 (2,-1,0,1,0) \}$$

$$= \text{span} \{ (2,1,1,0,0)\ (2,-1,0,1,0) \}$$

$$B = \{ (2,1,1,0,0)\ (2,-1,0,1,0) \}$$

(xiv) is $(3,6,1,0,2) \in Z(L)$?

$$Z(3,6,1,0,2) = c_1\,(2,1,1,0,0) + c_2\,(2,-1,0,1,0)$$

$= 2c_1 + 2c_2 = 3$

$c_1 - c_2 = 6$

$c_1 = 1$

$c_2 = 0$ impossible $\therefore$ NO it doesn't $\in Z(L)$

$0 = 2$

Q2: Let $T: \mathbb{R}^4 \to \mathbb{R}^3$ such that $T(x_1\ x_2\ x_3\ x_4) =$

$$\begin{aligned}(\ & 2x_1 - x_3 + x_4, \\ & -6x_2, \\ & 4x_1 - 6x_2 - 2x_3 + 2x_4\ )\end{aligned}$$

(i) find the standard matrix presentation of T

$$\{x_1\,(2,0,4) + x_2\,(0,-6,-6) + x_3\,(-1,0,-2) + x_4\,(1,0,2)\}$$

$$M = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & -6 & 0 & 0 \\ 4 & -6 & -2 & 2 \end{bmatrix}$$

(ii) find the standard basis of $\mathbb{R}^4$ and clearly state the relation between the standard matrix of $\mathbb{R}^4$ (domain) and m.

Standard basis of $\mathbb{R}^4$ = $\{ \underset{e_1}{(1,0,0,0)} \ \underset{e_2}{(0,1,0,0)} \ \underset{e_3}{(0,0,1,0)} \ \underset{e_4}{(0,0,0,1)} \}$

$T(e_1) = (2,0,4)$ $T(e_2) = (0,-6,-6)$

$T(e_3) = (-1,0,-2)$ $T(e_4) = (1,0,2)$

(iii) find $Z(T)$ and write it as span of basis

$$m = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & -6 & 0 & 0 \\ 4 & -6 & -2 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -6 & 0 & 0 \\ 4 & -6 & -2 & 2 \end{bmatrix} \underset{\sim}{-4R_1+R_3 \to R_3}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -6 & 0 & 0 \\ 0 & -6 & 0 & 0 \end{bmatrix} \underset{\sim}{-\frac{1}{6}R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & -6 & 0 & 0 \end{bmatrix} \underset{\sim}{6R_2+R_3 \to R_3}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

$\dim(Z(T))$ = number of free variables = 2

$x_1 = \frac{1}{2}x_3 - \frac{1}{2}x_4$
$x_2 = 0$

$= \{ (\frac{1}{2}x_3 - \frac{1}{2}x_4, 0, x_3, x_4) \mid x_3\ x_4 \in \mathbb{R} \}$

$= \{ x_3(\frac{1}{2}, 0, 1, 0) + x_4(-\frac{1}{2}, 0, 0, 1) \}$

$= \text{Span}\{ (\frac{1}{2}, 0, 1, 0)\ (-\frac{1}{2}, 0, 0, 1) \}$

$B = \{ (\frac{1}{2}, 0, 1, 0)\ (-\frac{1}{2}, 0, 0, 1) \}$

(iv) find Range (T) and write it as span of basis

$\text{Range}(T) = \text{Col}(m) = \text{Span}\{(2,0,4)\ (0,-6,-6)\}$

(v) is $(6,-12,0) \in \text{Range}(T)$?

$(6,-12,0) = c_1(2,0,4) + c_2(0,-6,-6)$

$2c_1 = 6 \qquad c_1 = 6/2 = 3$

$-6c_2 = -12 \qquad c_2 = -12/-6 = 2$

$4c_1 - 6c_2 = 0 \qquad 4(3) - 6(2) = 0$

$0 = 0$

(vi) find all the points in the domain ($\mathbb{R}^4$) such that

$T(x_1, x_2, x_3, x_4) = (6,-12,0)$

$$\left[\begin{array}{cccc|c} 2 & 0 & -1 & 1 & 6 \\ 0 & -6 & 0 & 0 & -12 \\ 4 & -6 & -2 & 2 & 0 \end{array}\right] \underset{\sim}{\tfrac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & 3 \\ 0 & -6 & 0 & 0 & -12 \\ 4 & -6 & -2 & 2 & 0 \end{array}\right] \underset{\sim}{-4R_1+R_3 \to R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & 3 \\ 0 & -6 & 0 & 0 & -12 \\ 0 & -6 & 0 & 0 & -12 \end{array}\right] \underset{\sim}{-\tfrac{1}{6}R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & -6 & 0 & 0 & -12 \end{array}\right] \underset{\sim}{6R_2+R_3 \to R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - 1/2\, x_3 + 1/2\, x_4 = 3 \rightarrow x_1 = 1/2\, x_3 - 1/2\, x_4 + 3$

$x_2 = 2$

$0 = 0$

sol doesn't have this

$= \{(1/2\, x_3 - 1/2\, x_4 + 3,\ 2,\ x_3,\ x_4) \mid x_3\ x_4 \in \mathbb{R}\}$

(vii) is T ONTO ?

$\dim(\text{Range}(T)) = \text{codomain}$

$2 \neq 3$

T is not ONTO

(viii) is T one-to-one?

IF $Z(T) = \{$ origin of the domain $\} = \{(0,0,0,0)\}$

T is not one-to-one since

$Z(T) = \text{span}\{(1/2, 0, 1, 0)(-1/2, 0, 0, 1)\}$

(ix) is T an isomorphism?

no since its neither onto nor 1-1.

Q3: given $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$T(2,0,0) = (4,-2)$

$T(0,6,0) = (18,-6)$

$T(1,1,1) = (10,-5)$

(i) find the standard matrix presentation of T, m.

$T(e_i) =$ ith col of m

were taking it back to the origin

$T(2,0,0) = \frac{1}{2}T(2,0,0) = \frac{1}{2}(4,-2) = (2,-1)$

$T(0,6,0) = \frac{1}{6}T(0,6,0) = \frac{1}{6}(18,-6) = (3,-1)$

one of the standard basis

$T(1,1,1) = (10,-5)$

so, $T(1,1,1) - T(1,0,0) - T(0,1,0)$

$= (10,-5) - (2,-1) - (3,-1)$

$T(0,0,1) = (5,-3)$

$$m = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -1 & -3 \end{bmatrix}$$

(ii) find $T(4,-6,5)$

$$= m\begin{bmatrix}4\\-6\\5\end{bmatrix}$$

$$= \begin{bmatrix}2 & 3 & 5\\-1 & -1 & -3\end{bmatrix}\begin{bmatrix}4\\-6\\5\end{bmatrix}$$

$$= 4\begin{bmatrix}2\\-1\end{bmatrix} - 6\begin{bmatrix}3\\-1\end{bmatrix} + 5\begin{bmatrix}5\\-3\end{bmatrix}$$

$$= \left(\left[4(2)-6(3)+5(5)\right], \left[4(-1)-6(-1)+5(-3)\right]\right)$$

$$= \left(\left[\underbrace{8-18}_{10}+25\right], \left[\underbrace{-4+6}_{2}-15\right]\right)$$

$$= (35, -13)$$

system of linear equations:

$$\begin{array}{c} \\ \\ \end{array}\left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \underline{1} & 3 & 0 & 0 & 1\\ 0 & 0 & \underline{1} & 0 & 2\\ 0 & 0 & 0 & \underline{1} & 3\end{array}\right] \rightarrow \text{last step in calculation}$$

consistent, we have three leading variables

system of LE

$n \times m$

# of eq — # of variables

3 Possibilities:

(1) unique solution

(2) no solution

(3) infinitely many solutions

- if the system has (1) or (3), then the system is consistent
- if the system has (2), then we say its inconsistent

$x_1 + 3x_2 = 1$ — x_1 leading

$x_3 = 2$ — x_3 "

$x_4 = 3$ — x_4 "

$x_2 \in \mathbb{R}$

note: we must have at least one free variable

$x_1 = 1 - 3x_2$

1 free variable so consistent

$f = \{(1-3x_2, x_2, 2, 3) \mid x_2 \in \mathbb{R}\}$

is $(1, 0, 2, 3)$ belongs to f ?

we take $x_2 = 0$

yes, it belongs since its not a subspace

$x_2 = -2$
$(7, -2, 2, 3)$ ✓ belongs

$x_2 = 3$
$(8, 3, 2, 3)$ ✓ belongs

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

we have three leading variables
no free variables
inconsistent
so NO SOLUTION

system of LE has no solution IF in one of the steps

$$\underset{\text{augmented}}{[\quad]} \sim [\quad]$$

• you observe that one of the equations become
0 = non zero number

3 x 3 system of L.E.

$$x_1 + x_2 - x_3 = 1$$
$$-x_1 + 2x_3 = 2$$
$$2x_1 + 3x_2 - 2x_3 = 10$$

write it in augmented matrix

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -1 & 0 & 2 & 2 \\ 2 & 3 & -2 & 10 \end{array}\right] \end{array} \underset{-2R_1 + R_3 \to R_3}{\overset{R_1 + R_2 \to R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 8 \end{array}\right]$$

$$\underset{-R_2 + R_3 \to R_3}{\overset{-R_2 + R_1 \to R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & 5 \end{array}\right] \underset{-R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$\underset{-R_3 + R_2 \to R_2}{\overset{2R_3 + R_1 \to R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

the set consists of one point
$(x_1, x_2, x_3) = (-2, 8, -5)$

$-x_1 + 2x_2 - 3x_3 = 4$
$-x_1 + a x_2 + 5x_3 = 10$
$2x_1 + 4x_2 + b x_3 = c$

① for what values of a, b, c does the system have unique solution ?
② " " " " " will " " be inconsistent ?
③ " " " " " will " " have infinitely many sol ?

$$\begin{array}{c} \quad x_1 \quad x_2 \quad x_3 \\ \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ -1 & a & 5 & 10 \\ 2 & 4 & b & c \end{array}\right] \end{array} \begin{array}{c} R_1 + R_2 \to R_2 \\ \sim \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & a+2 & 2 & 14 \\ 0 & 0 & b+6 & c-8 \end{array}\right]$$

you cant do anymore row op

$x_1 + 2x_2 - 3x_3 = 4$
$(a+2)x_2 + 2x_3 = 14$
$(b+6)x_3 = c-8$

unique sol: $a \neq -2 \quad b \neq -6 \quad c \in \mathbb{R}$

inconsistent "0 = not zero"
$b = -6 \qquad c \neq 8$

$b \neq -6 \qquad x_3 = \dfrac{c-8}{b+6} = 7$

$\therefore \; a = -2 \;\&\; \dfrac{c-8}{b+6} \neq 7$

infinitely many sol:
which is when we have at least 1 free variable

- $a = -2$ ∴ x_2 will become a free variable
  $\dfrac{c-8}{b+6} = 7$ $\quad x_3 = 7$ for it to be consistent
- $b = -6$ & $c = 8$ $\quad 0 = 0$ for the x_3 case $\quad a \neq -2$

$T: \mathbb{R}^4 \to \mathbb{R}^4$
$T(x_1\ x_2\ x_3\ x_4) = (4x_1, -2x_2, 3x_3, -x_4)$ → eigen value review

$\alpha = 4 \qquad T(1,0,0,0) = (4,0,0,0) = 4(1,0,0,0)$ ∴ 4 is eigen value

eigen space corresponds to the eigen value 4

subspace of the domain (here $\mathbb{R}^4$)

$E_4 = \text{span}\{(1,0,0,0)\}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 6 \end{bmatrix}$$

find all eigen values of A. for each eigenvalue of A, say α, find E_α.

note: we only study eigen for $n \times n$ matrices

meaning find real number, α, such that there exists at least one point in $\mathbb{R}^3$, say $Q = (x_1\ x_2\ x_3) \neq (0,0,0)$

s.t. $\underset{3\times3}{A} \underset{3\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Tools needed to find eigen values:

(1) determinant

$$A = \begin{array}{c} (1,1) \\ \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ 1 & 2 & 6 \end{bmatrix} \end{array}$$

find $|A|$. ← means determinant

choose any row or any col (recommended we choose the one that has more zeros)

1st col:

$$(-1)^{\overset{\text{loc}}{1+1}}(1)\begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} + (-1)^{2+1}(2)\begin{vmatrix} 3 & -1 \\ 2 & 6 \end{vmatrix} + (-1)^{3+1}(1)\begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix}$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD \underset{\text{minus}}{-} BC$$

$$= (4)(6) - 2 - 2(18+2) + 1(3+4) = -11$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 4 & 3 \\ 0 & 6 & 10 \end{bmatrix}$$

$$= (-1)^{3+2}(6)\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + (-1)^{3+3}(10)\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= 6(3-0) + 10(4-2) = -18 + 20 = 2$$

facts about determinant:

(1) $n \times n$ system of linear eq.

$$\left[\begin{array}{c|c} \underbrace{x_1 \cdots x_n}_{\text{coeff matrix}} & c \end{array}\right] = \left[\begin{array}{c|c} C & c \end{array}\right]$$

has unique solution IF the determinant of $|C| \neq 0$

(2) IF $|C| = 0$ then, → no solution
→ infinitely many

Cramer- Rule :

explain by example:

$$\left.\begin{aligned} x_1 + 2x_2 - x_3 &= 10 \\ x_1 + 4x_2 + 10x_3 &= 11 \\ -3x_1 + 10x_2 + 9x_3 &= 30 \end{aligned}\right] \rightarrow 3 \times 3$$

$$\text{assume } |C| = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 2 & -1 \\ 1 & 4 & 10 \\ -3 & 10 & 9 \end{bmatrix} \end{array} \neq 0$$

$$x_1 = \frac{\begin{array}{c} \begin{matrix} C & x_2 & x_3 \end{matrix} \\ \begin{vmatrix} 10 & 2 & -1 \\ 11 & 4 & 10 \\ 30 & 10 & 9 \end{vmatrix} \end{array}}{|C|} \qquad x_2 = \frac{\begin{array}{c} \begin{matrix} x_1 & C & x_3 \end{matrix} \\ \begin{vmatrix} 1 & 10 & -1 \\ 1 & 11 & 10 \\ -3 & 30 & 9 \end{vmatrix} \end{array}}{|C|} \qquad x_3 = \frac{\begin{array}{c} \begin{matrix} x_1 & x_2 & C \end{matrix} \\ \begin{vmatrix} 1 & 2 & 10 \\ 1 & 4 & 11 \\ -3 & 10 & 30 \end{vmatrix} \end{array}}{|C|}$$

- the effect of row operation on $|A|$ (determinant)

Explain by doing an example :

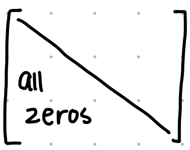
$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 2 & 5 \\ -1 & -2 & 10 \end{bmatrix} \quad \begin{matrix} 2R_1 + R_2 \rightarrow R_2 \\ \sim \\ R_1 + R_3 \rightarrow R_3 \end{matrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 11 \\ 0 & 0 & 13 \end{bmatrix} \quad \therefore \text{ upper triangular}$$

$$|B| = \det(B) = |A|$$
$$= (1)(6)(13) = 78$$

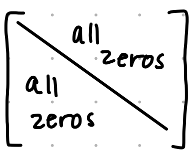
Result:

let A be $n \times n$ triangular matrix , then $|A| =$ multiplication of all numbers on the main diagonal

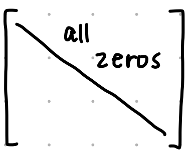
Def: A is triangular if it has one of the following forms:
has to be $n \times n$



= upper zeros



= diagonal



= lower zeros

$$\overset{\frac{1}{6}R_2}{\sim}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 11/6 \\ 0 & 0 & 13 \end{bmatrix}$$

$|C| = \frac{1}{6}|B| = \frac{1}{6}|A| = 13$

$\therefore |A| = 6|C| = 6(13) = 78$

Q: $A = \begin{bmatrix} 0 & \underline{4} & 12 \\ 1 & 0 & 10 \\ 4 & -2 & 6 \end{bmatrix}$ Find $|A|$

$$\overset{\frac{1}{4}R_1}{\sim}\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 10 \\ 4 & -2 & 6 \end{bmatrix} \overset{2R_1+R_3\to R_3}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 10 \\ 4 & 0 & 12 \end{bmatrix} \overset{-4R_2+R_3\to R_3}{\sim}$$

$|B| = \frac{1}{4}|A|$ $|C| = |B| = \frac{1}{4}|A|$

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 10 \\ 0 & 0 & -28 \end{bmatrix} \overset{R_1 \longleftrightarrow R_2}{} \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 3 \\ 0 & 0 & -28 \end{bmatrix}$$

$\therefore$ upper zeros

$|D| = |C| = \frac{1}{4}|A|$ $|E| = -|D| = -\frac{1}{4}|A|$

$|A| = (-4)|E| = (-4)(1)(1)(-28) = 112$

we multiply by a -ve when we interchange

$A = \begin{bmatrix} 2 & 4 & 6 & 10 \\ -2 & 5 & 10 & 13 \\ -4 & -8 & 10 & 10 \\ 16 & 32 & 48 & 100 \end{bmatrix}$ 4×4 $n\times n$ ✓

$$\overset{\frac{1}{2}R_1}{\sim}\begin{bmatrix} 1 & 2 & 3 & 5 \\ -2 & 5 & 10 & 13 \\ -4 & -8 & 10 & 10 \\ 16 & 32 & 48 & 100 \end{bmatrix}$$

$2R_1 + R_2 \to R_2$
$\sim$
$4R_1 + R_3 \to R_3$
$-16R_1 + R_4 \to R_4$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 9 & 16 & 23 \\ 0 & 0 & 22 & - \\ 0 & 0 & 0 & 20 \end{bmatrix}$$

$|B| = \frac{1}{2}|A|$ $|C| = \frac{1}{2}|A|$

$|A| = 2|C| = (2)(1)(9)(22)(20)$
$= \#$

Big Result:

A, B are n x n matrices

(1) $|AB| = |A||B|$ in particular, $|A^{m}| = [\,|A|\,]^m$

+ve integer

$\underbrace{A \times A \times A \cdots \times A}_{m\text{-times}}$

(2) $|\alpha A| = \alpha^n |A|$

scalar

(3) $|A^T| = |A|$

↳ Def: A, n x m

$$A^T = \begin{bmatrix} 1^{st} \text{ col of } A \\ \cdots \\ m^{th} \text{ col of } A \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \end{bmatrix}$ 2 x 3 $\quad A^T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ 3 x 2

in general, $(\underset{n\times m}{A}, \underset{m\times n}{B})$

$\boxed{AB}$ need not equal to $\boxed{BA}$

(4) $|AB| = |BA|$

(5) in general, $|A \pm B|$ need not equal to $|A| \pm |B|$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

$|A| = 0$ $\quad |B| = 0$

$|A| + |B| = 0$

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad |A+B| = 3 \neq |A| + |B|$

small result but useful:

I_n = identity matrix n x n

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad |I| = 1$

whenever multiplication is legal

$I_n B = B$

$B I_n = B$

$\underset{3\times 5}{A}\ \underset{5\times 5}{I_5} = A$

$\underset{3\times 3}{I_3}\ \underset{3\times 5}{A} = A$

A, $n \times n$

imagine α is an eigen value of A

$\Rightarrow \exists$ non-zero point $(a_1, \cdots, a_n) \in \mathbb{R}^n$

s.t. $$A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha I_n \underset{n \times 1}{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}$$

$$\alpha I_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} - A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha I_n - A \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$|\alpha I_n - A| = 0$$

Q: $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}_{2 \times 2}$

find all the values of A

set $|\alpha I_2 - A| = 0$, solve for α

characteristic polynomial of A, $(A) = |\alpha I_2 - A|$

$$= \left| \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \alpha - 1 & -2 \\ 0 & \alpha - 4 \end{vmatrix} = 0$$

$$(\alpha - 1)(\alpha - 4) = 0$$

$\therefore \alpha = 1 \quad \alpha = 4$

Questions WS:

Q1: find the solution set of the following 4x5 system of LE.

$$x_1 - x_2 + 2x_3 - x_4 + 4x_5 = 8$$

$$-x_1 + x_2 - x_3 + 4x_4 + x_5 = 2$$

$$-2x_1 + 2x_2 - 3x_3 + 5x_4 - 3x_5 = -6$$

$$3x_1 - 3x_2 + 6x_3 - 3x_4 + 12x_5 = 24$$

create the aug. matrix

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & c \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & -1 & 4 & 8 \\ -1 & 1 & -1 & 4 & 1 & 2 \\ -2 & 2 & -3 & 5 & -3 & -6 \\ 3 & -3 & 6 & -3 & 12 & 24 \end{array}\right] \underset{\substack{2R_1+R_3 \to R_3 \\ -3R_1+R_4 \to R_4}}{\overset{R_1+R_2 \to R_2}{\sim}} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & -1 & 4 & 8 \\ 0 & 0 & 1 & 3 & 5 & 10 \\ 0 & 0 & 1 & 3 & 5 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \underset{-R_2+R_3 \to R_3}{\overset{-2R_2+R_1 \to R_1}{\sim}}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & -7 & 6 & -12 \\ 0 & 0 & 1 & 3 & 5 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rightarrow x_1 - x_2 - 7x_4 + 6x_5 = -12 \rightarrow x_1 = x_2 + 7x_4 - 6x_5 - 12$

$\rightarrow x_3 + 3x_4 + 5x_5 = 10 \rightarrow x_3 = -3x_4 - 5x_5 + 10$

$\therefore$ leading variables are x_1 & x_3

$x_2, x_4, x_5 \approx$ free variables

$$= \{(x_2 + 7x_4 - 6x_5 - 12,\ x_2,\ -3x_4 - 5x_5 + 10,\ x_4,\ x_5) \mid x_2, x_4, x_5 \in \mathbb{R}\}$$

the solution set is not a subspace of $\mathbb{R}^5$

we have infinitely many solutions and each sol. is a point in $\mathbb{R}^5$

Q2: consider the above system, but make all constants zeros. Note that if all const. are zeros, then the system is called homogeneous system.

$$x_1 - x_2 + 2x_3 - x_4 + 4x_5 = 0$$
$$-x_1 + x_2 - x_3 + 4x_4 + x_5 = 0$$
$$-2x_1 + 2x_2 - 3x_3 + 5x_4 - 3x_5 = 0$$
$$3x_1 - 3x_2 + 6x_3 - 3x_4 + 12x_5 = 0$$

find the solution set. is the solution set a subspace of $\mathbb{R}^5$, if yes, then write it as a span and find dim (Solution - set)

$$
\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & c
\end{array}
$$

$$
\left[\begin{array}{ccccc|c}
1 & -1 & 2 & -1 & 4 & 0 \\
-1 & 1 & -1 & 4 & 1 & 0 \\
-2 & 2 & -3 & 5 & -3 & 0 \\
3 & -3 & 6 & -3 & 12 & 0
\end{array}\right]
\begin{array}{l}
R_1 + R_2 \to R_2 \\
\sim \\
2R_1 + R_3 \to R_3 \\
-3R_1 + R_4 \to R_4
\end{array}
\left[\begin{array}{ccccc|c}
1 & -1 & 2 & -1 & 4 & 0 \\
0 & 0 & 1 & 3 & 5 & 0 \\
0 & 0 & 1 & 3 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{array}{l}
-2R_2 + R_1 \to R_1 \\
\sim \\
-R_2 + R_3 \to R_3
\end{array}
$$

$$
\left[\begin{array}{ccccc|c}
1 & -1 & 0 & -7 & 6 & 0 \\
0 & 0 & 1 & 3 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{array}{l}
\to \; x_1 - x_2 - 7x_4 + 6x_5 = 0 \;\to\; x_1 = x_2 + 7x_4 - 6x_5 \\
\to \; x_3 + 3x_4 + 5x_5 = 0 \;\to\; x_3 = -3x_4 - 5x_5
\end{array}
$$

x_1 & x_3 leading variables

x_2, x_4, x_5 free variables

$$= \{ (x_2 + 7x_4 - 6x_5,\ x_2,\ -3x_4 - 5x_5,\ x_4,\ x_5) \mid x_2, x_4, x_5 \in \mathbb{R} \}$$

$$= \{ x_2(1,1,0,0,0) + x_4(7,0,-3,1,0) + x_5(-6,0,-5,0,1) \}$$

$$= \operatorname{Span}\{(1,1,0,0,0)\ (7,0,-3,1,0)\ (-6,0,-5,0,1)\}$$

$$\dim(\text{sol. set}) = 3$$

it is always true, the solution set of homogeneous system of linear equations is a subspace and dim (sol - set) = number of free variables

IMPORTANT DISCUSSION:

Let $T: \mathbb{R}^5 \to \mathbb{R}^4$ s.t. $T(x_1\, x_2\, x_3\, x_4\, x_5) =$

$$
\begin{aligned}
(\, & x_1 - x_2 + 2x_3 - x_4 + 4x_5, \\
& -x_1 + x_2 - x_3 + 4x_4 + x_5, \\
& -2x_1 + 2x_2 - 3x_3 + 5x_4 - 3x_5, \\
& 3x_1 - 3x_2 + 6x_3 - 3x_4 + 12x_5 \,)
\end{aligned}
$$

(i) if you find the standard matrix presentation of T, M, then the M is the aug matrix.

(ii) the points in the domain of $T(\mathbb{R}^5)$ is the solution set of the system.

(iii) $Z(T) = \operatorname{Ker}(T) = \operatorname{Null}(T)$ is the sol. set of the homogeneous system $[M \mid 0]$

i.e. $Z(T) = \operatorname{Span}\{(1,1,0,0,0)\ (7,0,-3,1,0)\ (-6,0,-5,0,1)\}$

Q3: find the solution set of the following system:

$$2x_2 + 4x_3 + 8x_4 = 10$$
$$x_1 - x_2 + 2x_3 - 4x_4 = -7$$
$$2x_1 + 8x_3 + 2x_4 = 12$$

$$\left[\begin{array}{cccc|c} 0 & 2 & 4 & 8 & 10 \\ 1 & -1 & 2 & -4 & -7 \\ 2 & 0 & 8 & 2 & 12 \end{array}\right] \underset{\sim}{\tfrac{1}{2}R_1} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 4 & 5 \\ 1 & -1 & 2 & -4 & -7 \\ 2 & 0 & 8 & 2 & 12 \end{array}\right] \underset{\sim}{R_1 + R_2 \to R_2}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 4 & 5 \\ 1 & 0 & 4 & 0 & -2 \\ 2 & 0 & 8 & 2 & 12 \end{array}\right] \underset{\sim}{-2R_2 + R_3 \to R_3} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 4 & 5 \\ 1 & 0 & 4 & 0 & -2 \\ 0 & 0 & 0 & 2 & 16 \end{array}\right]$$

$x_2 + 2x_3 + 4x_4 = 5$
$x_2 = -2x_3 - 4x_4 + 5$
$= -2x_3 - 4(8) + 5$
$= -2x_3 - 27$

$x_1 + 4x_3 = -2$
$x_1 = -4x_3 - 2$

$2x_4 = 16$
$x_4 = 16/2 = 8$

x_2, x_1, x_4 leading variables
x_3 free variables

$= \{(-4x_3 - 2, -2x_3 - 27, x_3, 8) \mid x_3 \in \mathbb{R}\}$

Q4: find the solution set of the homogeneous system

$$2x_2 + 4x_3 + 8x_4 = 0$$
$$x_1 - x_2 + 2x_3 - 4x_4 = 0$$
$$2x_1 + 8x_3 + 2x_4 = 0$$

$$\left[\begin{array}{cccc|c} 0 & 2 & 4 & 8 & 0 \\ 1 & -1 & 2 & -4 & 0 \\ 2 & 0 & 8 & 2 & 0 \end{array}\right] \underset{\sim}{\tfrac{1}{2}R_1} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 4 & 0 \\ 1 & -1 & 2 & -4 & 0 \\ 2 & 0 & 8 & 2 & 0 \end{array}\right] \underset{\sim}{R_1 + R_2 \to R_2}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 4 & 0 \\ 1 & 0 & 4 & 0 & 0 \\ 2 & 0 & 8 & 2 & 0 \end{array}\right] \underset{\sim}{-2R_2 + R_3 \to R_3} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 4 & 0 \\ 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$x_2 + 2x_3 + 4x_4 = 0$
$x_2 = -2x_3 - 4x_4$
$= -2x_3 - 4(0)$
$= -2x_3$

$x_1 + 4x_3 = 0$
$x_1 = -4x_3$

$2x_4 = 0$
$x_4 = 0$

$= \{(-4x_3, -2x_3, x_3, 0) \mid x_3 \in \mathbb{R}\}$

$= \{x_3(-4, -2, 1, 0)\} = \text{span}\{(-4, -2, 1, 0)\}$

Q5: find the solution set of the following system

$$2x_2 + 4x_3 + 8x_4 = 10$$
$$x_1 - x_2 + 2x_3 + 2x_4 = -7$$
$$2x_1 + 8x_3 + 12x_4 = 12$$

$$\left[\begin{array}{cccc|c} 0 & 2 & 4 & 8 & 10 \\ 1 & -1 & 2 & 2 & -7 \\ 2 & 0 & 8 & 12 & 12 \end{array}\right] \underset{\sim}{\tfrac{1}{2}R_1} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 4 & 5 \\ 1 & -1 & 2 & 2 & -7 \\ 2 & 0 & 8 & 12 & 12 \end{array}\right] \underset{\sim}{R_1 + R_2 \to R_2}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 4 & 5 \\ 1 & 0 & 4 & 6 & -2 \\ 2 & 0 & 8 & 12 & 12 \end{array}\right] \underset{-2R_2+R_3}{\sim} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 4 & 5 \\ 1 & 0 & 4 & 6 & -2 \\ 0 & 0 & 0 & 0 & 16 \end{array}\right] \quad 0 \neq 16$$

$\therefore$ there are no solutions

Q6: find the solution set of the following system:

$$x_1 + 2x_2 + 4x_3 + 8x_4 = 10$$
$$x_1 - x_2 + 2x_3 + 2x_4 = -7$$
$$x_1 + 4x_2 + 8x_3 + 12x_4 = 12$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 10 \\ 1 & -1 & 2 & 2 & -7 \\ 1 & 4 & 8 & 12 & 12 \end{array}\right] \begin{array}{c} -R_1 + R_2 \to R_2 \\ \sim \\ -R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 10 \\ 0 & -3 & -2 & -6 & -17 \\ 0 & 2 & 4 & 4 & 2 \end{array}\right] \underset{\sim}{-\tfrac{1}{3}R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 10 \\ 0 & 1 & 2/3 & 2 & 17/3 \\ 0 & 2 & 4 & 4 & 2 \end{array}\right] \begin{array}{c} -2R_2 + R_1 \to R_1 \\ \sim \\ -2R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 8/3 & 4 & -4/3 \\ 0 & 1 & 2/3 & 2 & 17/3 \\ 0 & 0 & 8/3 & 0 & -28/3 \end{array}\right] \times 3/8\, R_3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 8/3 & 4 & -4/3 \\ 0 & 1 & 2/3 & 2 & 17/3 \\ 0 & 0 & 1 & 0 & -7/2 \end{array}\right]$$

$$x_1 + 8/3\, x_3 + 4x_4 = -4/3$$

$$x_1 = (-8/3)(-3.5) - 4/3 + 4x_4$$
$$= 8 + 4x_4$$

$$x_2 + 2/3\, x_3 + 2x_4 = 17/3$$
$$x_2 = (-2/3)(-3.5) + 17/3 + 2x_2$$
$$= 8 + 2x_2$$

$$x_3 = 3.5$$

Questions WS #5:

Q5: given $A = \begin{bmatrix} \boxed{2 \quad a \quad b} \\ c & 3 & 7 \\ d & -1 & 3 \end{bmatrix}$ and $|A| = -7$ where a b c & d are some #'s

(i) let $B = \begin{bmatrix} 10 & a & b \\ c & 3 & 7 \\ d & -1 & 3 \end{bmatrix}$ find $|B|$

$$|A| = (-1)^2 (2) \begin{vmatrix} 3 & 7 \\ -1 & 3 \end{vmatrix} + (-1)^3 (a) \begin{vmatrix} c & 7 \\ d & 3 \end{vmatrix} + (-1)^4 (b) \begin{vmatrix} c & 3 \\ d & -1 \end{vmatrix} = -7$$

$$= \underbrace{2\,(16)}_{32} - a\,(3c - 7d) + b\,(-c + 3d) = -7$$

$$= -a(3c - 7d) + b(-c + 3d) = -7 - 32$$

$$|B| = (-1)^2 (10) \begin{vmatrix} 3 & 7 \\ -1 & 3 \end{vmatrix} + (-1)^3 (a) \begin{vmatrix} c & 7 \\ d & 3 \end{vmatrix} + (-1)^4 (b) \begin{vmatrix} c & 3 \\ d & -1 \end{vmatrix}$$

$$= 10\,(16)\, \underbrace{-\, a\,(3c - 7d) + b\,(-c + 3d)}_{|A|}$$

$$= 160 - 7 - 32 = 121$$

(ii) let $C = \begin{bmatrix} 4 & 2a & 2b \\ d+2 & a-1 & b+3 \\ c & 3 & 7 \end{bmatrix}$ find $|C|$

$$\overset{A}{\begin{bmatrix} 2 & a & b \\ c & 3 & 7 \\ d & -1 & 3 \end{bmatrix}} \underset{\sim}{\times 2R_1} \overset{B}{\begin{bmatrix} 4 & 2a & 2b \\ c & 3 & 7 \\ d & -1 & 3 \end{bmatrix}} R_2 \longleftrightarrow R_3$$

$$\overset{D}{\begin{bmatrix} 4 & 2a & 2b \\ d & -1 & 3 \\ c & 3 & 7 \end{bmatrix}} \underset{\sim}{\tfrac{1}{2}R_1 + R_2} \overset{C}{\begin{bmatrix} 4 & 2a & 2b \\ d+2 & a-1 & b+3 \\ c & 3 & 7 \end{bmatrix}}$$

$|A| = -7$ $|B| = 2\,|A|$

$|D| = -2\,|A|$

$|C| = (-2)(-7) = 14$

Q6: find the solution set to the system:

$$x_1 + x_2 + x_3 + x_4 = 4$$
$$-x_1 - x_3 - x_4 = 10$$
$$-2x_1 - 2x_2 - 2x_3 - 2x_4 = -8$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ -1 & 0 & -1 & -1 & 10 \\ -2 & -2 & -2 & -2 & -8 \end{array}\right] \underset{2R_1+R_3}{\overset{R_1+R_2}{\sim}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 + x_3 + x_4 = 4$

$x_2 = 14$

$0 = 0$

$x_1 = -x_3 - x_4 + 4 - 14$ $\quad x_2 = 14$

$= -x_3 - x_4 - 10$

$= \{(-x_3 - x_4 - 10, 14, x_3, x_4) \mid x_3 x_4 \in \mathbb{R})\}$

Q7: given the augmented matrix of a system of LE $A = \left[\begin{array}{ccc|c} 1 & -a & 3 & 4 \\ -1 & 1+a & -3 & -2 \\ -1 & a & b & c \end{array}\right]$

(i) for what values of a, b, c will the system have a unique sol?

$$\left[\begin{array}{ccc|c} 1 & -a & 3 & 4 \\ -1 & 1+a & -3 & -2 \\ -1 & a & b & c \end{array}\right] \underset{R_1+R_3 \to R_3}{\overset{R_1+R_2 \to R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & -a & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & b+3 & c+4 \end{array}\right]$$

$a \in \mathbb{R}$ $\quad c \in \mathbb{R}$ $\quad b \neq -3$

(ii) for what values of a b c will the system have infinitely many sol?

$b = -3$ $\quad c = -4$ $\quad a \in \mathbb{R}$

Recall:

α is an eigen value of A

we know 2 things

(1) $|\alpha I_n - A| = 0$

(2) $\exists$ a non-zero point Q in $\mathbb{R}^n$, $(a_1 \cdots a_n)$ s.t.

$$\begin{bmatrix} \alpha I_n - A \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Q: $A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 2 & -4 & -5 \end{bmatrix}$

find all eigen values of A for each eigen value, α find $\underbrace{E_\alpha}_{\text{eigen space}}$

for (1) set $\underbrace{|\alpha I_3 - A|}_{\text{characteristic polynomial of A} \approx \text{char}(A)} = 0$ find α

$$\alpha I_3 - A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 2 & -4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ -2 & 4 & \alpha+5 \end{bmatrix} \underset{\sim}{R_2 + R_3 \to R_3} \begin{bmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ 0 & \alpha & \alpha \end{bmatrix}$$

you need this matrix to find E_α

$$\begin{vmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ 0 & \alpha & \alpha \end{vmatrix} = 0 \text{, solve for } \alpha$$

$$= (-1)^{3+2}(\alpha)\begin{vmatrix} \alpha-2 & -3 \\ 2 & -5 \end{vmatrix} + (-1)^{3+3}(\alpha)\begin{vmatrix} \alpha-2 & -1 \\ 2 & \alpha-4 \end{vmatrix} = 0$$

$$= -(\alpha)\left((\alpha-2)(-5) - (\underbrace{(-3)(2)}_{-6})\right) + (\alpha)\left((\alpha-2)(\alpha-4) - \underbrace{(-1)(2)}_{-2}\right) = 0$$

$$= (-\alpha)(-5\alpha + 10 + 6) + (\alpha)(\alpha^2 - 4\alpha - 2\alpha + 8 + 2)$$

$$= 5\alpha^2 - 16\alpha + \alpha^3 - 6\alpha^2 + 10\alpha$$

old exams
$IN(D) = \dim(D)$

$= \alpha^3 - \alpha^2 - 6\alpha = 0$

$= (\alpha)(\alpha^2 - \alpha - 6) = 0$

$= (\alpha)(\alpha-3)(\alpha+2) = 0$

$\alpha = 0 \quad \alpha = 3 \quad \alpha = -2$

$$\begin{bmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ 0 & \alpha & \alpha \end{bmatrix}$$

$\alpha = 0$

$E_{\alpha=0}$: Sol. set of the homogeneous system

$$\left[\begin{array}{c|c} 0\, I_3 - A & \begin{matrix} C \\ 0 \end{matrix} \end{array}\right]$$

the augmented matrix

$$\left[\begin{array}{ccc|c} -2 & -1 & -3 & 0 \\ 2 & -4 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{-\tfrac{1}{2}R_1} \left[\begin{array}{ccc|c} \underline{\underline{1}} & 1/2 & 3/2 & 0 \\ 2 & -4 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{\sim}{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 0 \\ 0 & \underline{\underline{-5}} & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{x -1/5 R_2} \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 0 \\ 0 & 1 & 8/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{\sim}{-1/2 R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 7/10 & 0 \\ 0 & 1 & 8/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x_1 + 7/10\, x_3 = 0 \\ \to x_2 + 8/5\, x_3 = 0 \\ \to 0 = 0 \end{array}$$

$\therefore$ leading var x_1 & x_2

free var x_3

$x_1 = -7/10\, x_3$

$x_2 = -8/5\, x_3$

$x_3 \in \mathbb{R}$

$E_0 = \{ (-7/10\, x_3, -8/5\, x_3, x_3) \mid x_3 \in \mathbb{R} \}$

$= \text{span} \{ (-7/10, -8/5, 1) \}$

E_α is a set of $\underline{\underline{\text{all}}}$ points in $\mathbb{R}^n$, say $Q = (a_1 \cdots a_n)$
where $A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

E_3 = Augmented matrix

$\alpha I_3 - A$

$$\begin{bmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ 0 & \alpha & \alpha \end{bmatrix} \quad \underline{\underline{so}} \quad \left[\begin{array}{ccc|c} -2+3 & -1 & -3 & 0 \\ 2 & 3-4 & -5 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 2 & -1 & -5 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \underset{\sim}{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$$\begin{matrix} R_2+R_1\to R_1 \\ \sim \\ -3R_2+R_3\to R_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 2x_3$

$x_2 = -x_3$

$x_3 \in \mathbb{R}$

$E_3 = \{ (2x_3, -x_3, x_3) \mid x_3 \in \mathbb{R} \}$

$= \text{span} \{ (2,-1,1) \}$

E_{-2} (it should be span of 1 point)

$D = \{ (x_1 \ x_2 \ x_3 \ x_4) \mid \begin{matrix} x_1 + x_2 - x_3 + x_4 = 0 \\ x_2 - 7x_3 - 2x_4 = 0 \end{matrix} \}$ is this a subspace?

yes its a homogeneous

to find the span you do the aug matrix etc

you'll get a span of 2 points

Questions WS#5:

<u>Q9:</u> given $D = \text{span} \{ (1,1,1,1) (-1,0,0,0) (0,1,1,1) (-1,1,1,1) \}$

(1) find $\dim(D) = 2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} R_1+R_2\to R_2 \\ \sim \\ R_1+R_4\to R_4 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix} \begin{matrix} -R_2+R_1\to R_1 \\ \sim \\ -R_2+R_3\to R_3 \\ -2R_2+R_4\to R_4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(2) in view of your answer to part (1), does $D = \mathbb{R}^4$? why ?

$D \neq \mathbb{R}^4$ because only two points are independent and for it to be in $\mathbb{R}^4$, it needs 4 indep. points

(3) convince me that the point $(2,8,8,8)$ lives inside D

$$D = \text{span}\{(1,1,1,1)\ (-1,0,0,0)\}$$

$$(2,8,8,8) = c_1(1,1,1,1) + c_2(-1,0,0,0)$$
$$= (c_1, c_1, c_1, c_1) + (-c_2, 0, 0, 0)$$

$2 = c_1 - c_2$
$8 = c_1$
$8 = c_1$
$8 = c_1$

$\therefore\ 2 = 8 - c_2$
$c_2 + 2 = 8$
$\underbrace{c_2 = 8 - 2 = 6}_{\text{here something is wrong he got } -2}$

linear comb:

$$0(1,1,1,1) - 2(-1,0,0,0) + 8(0,1,1,1) + 0(-1,1,1,1)$$
$$= (2,0,0,0) + (0,8,8,8) = (2,8,8,8)$$

(4) Does the point $(2,5,6,6)$ live in D ? explain

$$(2,5,6,6) = c_1(1,1,1,1) + c_2(-1,0,0,0)$$

$2 = c_1 - c_2$ $\quad c_2 + 2 = 5 \quad c_2 = 5 - 2 = 3$

$5 = c_1$

$$(2,5,6,6) \neq 5(1,1,1,1) + 3(-1,0,0,0)$$

cant be written as a linear combination

Q4: given $A = \begin{bmatrix} 1 & a & b & 4 \\ 2 & 4 & c & 0 \\ 0 & d & -9 & 3 \\ 1 & -2 & 3 & 1 \end{bmatrix}$ is equivalent to the matrix

$B = \begin{bmatrix} 1 & c & f & h \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ find the values of a, b, c, d

So, we know Rank(A) = 2 from B

$$(a,4,d,-2) = c_1(1,2,0,1) + c_2(4,0,3,1)$$

$$= (c_1, 2c_1, 0, c_1) + (4c_2, 0, 3c_2, c_2)$$

$$a = c_1 + 4c_2 = 2 + 4(-4) = 2 - 16 = -14$$

$$4 = 2c_1 \quad \therefore \; c_1 = 4/2 = 2$$

$$d = 3c_2 = (3)(-4) = -12$$

$$-2 = c_1 + c_2$$
$$-2 = 2 + c_2$$
$$c_2 = -4$$

$\therefore$ $a = -14$
$d = -12$

$$(b, c, -9, 3) = c_1(1,2,0,1) + c_2(4,0,3,1)$$
$$= c_1, \; 2c_1, \; 0, \; c_1$$
$$+ \; 4c_2 \quad 0 \quad 3c_2 \quad c_2$$

$b = c_1 + 4c_2$
$= 6 + 4(-3)$
$= -6$

$c = 2c_1$
$c = (2)(6)$
$= 12$

$-9 = 3c_2$
$c_2 = -9/3$
$= -3$

$3 = c_1 + c_2$
$3 = c_1 - 3$
$c_1 = 3 + 3 = 6$

Q5: given A is a 3×3 matrix s.t. 2 is an eigenvalue of A and $E_2 = \text{span}\{(1,2,-1)\,(0,-1,-4)\}$

(i) can we conclude that $A\begin{bmatrix} 3 \\ 4 \\ -11 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ -22 \end{bmatrix}$? explain

$$2\begin{bmatrix} 3 \\ 4 \\ -11 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ -22 \end{bmatrix} \quad \text{yes}$$

(ii) if A is diagnolizable and Trace $(A) = 4$. Find Rank (A). is A invertible (nonsingular). Explain.

Questions WS:

Determinant Eigenvalues
Eigenvectors

Q1: given A, B are 2x2 matrices s.t. $|A| = 2$ and $|B| = -3$

(i) find $|A^3 B^T|$

$$= |A|^3 \; |B^T| = (2)^3(-3) = (8)(-3) = -24$$

(ii) find $|A+B|$

$|A+B| \neq |A| + |B|$ ∴ we need more info.

(iii) find $|3BA|$

$$|cA| = c^n \; |A| \qquad n=2$$

$$= 3^2 \; (2)(-3) = (9)(2)(-3) = -54$$

(iv) consider the system of LE $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2.5 \\ 7 \end{bmatrix}$ what can you say about the solution? unique? infinite? undecided

its a unique solution because there is only value for x and one for y

$$|A| = 2 \neq 0$$

(v) let C be the second column of B. Find the solution set to the system $B\begin{bmatrix} x \\ y \end{bmatrix} = C$

$|B| = -3 \neq 0$ ∴ unique solution

$C =$ 2nd col of $|B| \approx \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$C = \{(0,1)\}$$

Q2: let A be a 3×4 matrix, and C be the third col of A.

Consider the system of linear equations $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = C$. Convince me that the system is consistent and it has infinitley many solutions.

aug matrix

$$[A \mid C]$$

we cant use determinants since its not $n\times n$
$\therefore$ we need to show that the system has a solution.

so

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1(1^{st}\text{ col } A) + x_2(2^{nd}\text{ col } A) + x_3(3^{rd}\text{ col } A) + x_4(4^{th}\text{ col } A)$$

$$= C\ (3^{rd}\text{ col of } A) = (0,0,1,0)$$

$\therefore$ the sol. set is consistent

A is $\underset{\downarrow}{3} \times \underset{\downarrow}{4}$
3 eq 4 var

$\therefore$ we can conclude that we will have 3 leading variables at most and 1 free variable resulting in infinitley many sols.

Q3: let A be a 4×4 given:

$$A \xrightarrow{2R_4} B \xrightarrow{R_1 \leftrightarrow R_3} C \xrightarrow{2R_1+R_4\to R_4} D = \begin{bmatrix} \boxed{2} & 4 & 4 & 6 \\ \boxed{0} & 1 & 6 & 6 \\ \boxed{0} & 0 & 0 & 2 \\ \boxed{0} & -1 & 2 & 1 \end{bmatrix}$$

find $|A|$.

$$|B| = 2|A|$$

$$|C| = -|B|$$

$$|D| = |C| = -2|A|$$

$$D = \begin{bmatrix} 2 & 4 & 4 & 6 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 0 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix} \underset{\sim}{R_2+R_4\to R_4} \begin{bmatrix} 2 & 4 & 4 & 6 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 8 & 7 \end{bmatrix} \underset{\sim}{R_3 \leftrightarrow R_4} \begin{bmatrix} 2 & 4 & 4 & 6 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 8 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

upper triangular

$$|E| = -2|A|$$

$$|F| = 2|A|$$

$$|F| = (2)(1)(8)(2) = 32$$

$$32 = 2|A|$$

$$|A| = 32/2 = 16$$

Q4: consider the following system of LE.

$$x_1 + a x_2 + 7x_3 + b x_4 = 30$$
$$-x_1 + 8x_2 + x_3 + a x_4 = 20$$
$$-2x_1 - 2a x_2 + c x_3 + x_4 = 2$$
$$-x_1 - a x_2 - 7x_3 - 12 x_4 = -7$$

for what values of a b c will the system have a unique sol?

$$\left[\begin{array}{cccc|c} 1 & a & 7 & b & 30 \\ -1 & 8 & 1 & a & 20 \\ -2 & -2a & c & 1 & 2 \\ -1 & -a & -7 & -12 & -7 \end{array}\right] \underset{R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & a & 7 & b \\ 0 & a+8 & 8 & b+a \\ -2 & -2a & c & 1 \\ -1 & -a & -7 & -12 \end{bmatrix}$$

$$\underset{2R_1+R_3\to R_3}{\sim} \begin{bmatrix} 1 & a & 7 & b \\ 0 & a+8 & 8 & b+a \\ 0 & 0 & c+14 & 1+2b \\ -1 & -a & -7 & -12 \end{bmatrix} \underset{R_1+R_4\to R_4}{\sim} \begin{bmatrix} 1 & a & 7 & b \\ 0 & a+8 & 8 & b+a \\ 0 & 0 & c+14 & 1+2b \\ 0 & 0 & 0 & b-12 \end{bmatrix}$$

$$|A| = |D| = (1)(a+8)(c+14)(b-12)$$

$|A| \neq 0 \quad \therefore \; a \neq -8 \quad c \neq -14 \quad b \neq 12$

for us to have a unique sol.

Q5: Let $A = \begin{bmatrix} 1 & 2 & 10 \\ -1 & 4 & 5 \\ 1 & -4 & -5 \end{bmatrix}$

(i) find $C_A(\alpha)$ i.e. the characteristic polynomial of A

$$|\alpha I_n - A| = |\alpha I_3 - A|$$

$$= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 1 & 2 & 10 \\ -1 & 4 & 5 \\ 1 & -4 & -5 \end{bmatrix} = \begin{bmatrix} \alpha-1 & -2 & -10 \\ 1 & \alpha-4 & -5 \\ -1 & 4 & \alpha+5 \end{bmatrix} \underset{R_2+R_3\to R_3}{\sim} \begin{bmatrix} \alpha-1 & -2 & -10 \\ 1 & \alpha-4 & -5 \\ 0 & \alpha & \alpha \end{bmatrix}$$

$$= (-1)^5(\alpha)\begin{vmatrix} \alpha-1 & -10 \\ 1 & -5 \end{vmatrix} + (-1)^6(\alpha)\begin{vmatrix} \alpha-1 & -2 \\ 1 & \alpha-4 \end{vmatrix}$$

$$= -\alpha \left[(\alpha-1)(-5) + 10 \right] + \alpha \left[(\alpha-1)(\alpha-4) + 2 \right]$$

$$= -\alpha \left[-5\alpha + 5 + 10 \right] + \alpha \left[\alpha^2 - \alpha - 4\alpha + 4 + 2 \right]$$

$$= +\cancel{5\alpha^2} - 15\alpha + \alpha^3 - \cancel{5\alpha^2} + 6\alpha$$

$$= \alpha^3 - 9\alpha$$

(ii) find the eigen values of A.

$$= \alpha(\alpha^2 - 9)$$

$$\alpha = 0 \qquad \sqrt{\alpha^2} = \sqrt{9}$$

$$\alpha = \pm 3$$

(iii) for each eigenvalue of A, find the corresponding eigenspace and write it as span.

$$\alpha = 0, 3, -3$$

E_0 homog. system

$$\begin{bmatrix} \alpha-1 & -2 & -10 \\ 1 & \alpha-4 & -5 \\ -1 & 4 & \alpha+5 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & -10 \\ 1 & -4 & -5 \\ -1 & 4 & 5 \end{bmatrix} \underset{\sim}{-1R_1}$$

$$\begin{bmatrix} 1 & 2 & 10 \\ 1 & -4 & -5 \\ -1 & 4 & 5 \end{bmatrix} \underset{\sim}{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 10 \\ 0 & -6 & -15 \\ -1 & 4 & 5 \end{bmatrix} \underset{\sim}{-\tfrac{1}{6}R_2}$$

$$\begin{bmatrix} 1 & 2 & 10 \\ 0 & 1 & 5/2 \\ -1 & 4 & 5 \end{bmatrix} \underset{-4R_2+R_3 \to R_3}{\overset{-2R_2+R_1 \to R_1}{\sim}} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5/2 \\ -1 & 0 & -5 \end{bmatrix} \underset{-R_3}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5/2 \\ 1 & 0 & 5 \end{bmatrix} \underset{\sim}{-R_3+R_1 \to R_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 5/2 \\ 1 & 0 & 5 \end{bmatrix}$$

$$x_2 + 5/2\, x_3 = 0 \qquad x_2 = -5/2\, x_3$$

$$x_1 + 5x_3 = 0 \qquad x_1 = -5x_3$$

$$= \{ (-5x_3, -5/2\, x_3, x_3) \mid x_3 \in \mathbb{R} \}$$

$$= \text{span}\{ (-5, -5/2, 1) \} \qquad \dim(E_0) = 1$$

$$E_3 = \begin{bmatrix} 3-1 & -2 & -10 \\ 1 & 3-4 & -5 \\ -1 & 4 & 3+5 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & -10 \\ 1 & -1 & -5 \\ -1 & 4 & 8 \end{bmatrix} \begin{matrix} \tfrac{1}{2}R_1 \\ \sim \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & -5 \\ 1 & -1 & -5 \\ -1 & 4 & 8 \end{bmatrix} \begin{matrix} -R_1+R_2 \to R_2 \\ \sim \\ R_1+R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & -1 & -5 \\ 0 & 0 & 0 \\ 0 & 3 & 3 \end{bmatrix} \begin{matrix} \sim \\ \tfrac{1}{3}R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & -5 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \sim \\ R_3+R_1 \to R_1 \end{matrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} x_1 - 4x_3 = 0 & x_1 = 4x_3 \\ x_2 + x_3 = 0 & x_2 = -x_3 \end{matrix}$$

$$= \{(4x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\}$$

$$= \text{span}\{(4,-1,1)\} \qquad \dim(E_3) = 1$$

$$E_{-3} : \begin{bmatrix} -3-1 & -2 & -10 \\ 1 & -3-4 & -5 \\ -1 & 4 & -3+5 \end{bmatrix} \sim \begin{bmatrix} -4 & -2 & -10 \\ 1 & -7 & -5 \\ -1 & 4 & 2 \end{bmatrix} \begin{matrix} -\tfrac{1}{4}R_1 \\ \sim \end{matrix}$$

$$\begin{bmatrix} 1 & 1/2 & 5/2 \\ 1 & -7 & -5 \\ -1 & 4 & 2 \end{bmatrix} \begin{matrix} -R_1+R_2 \to R_2 \\ \sim \\ R_1+R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 1/2 & 5/2 \\ 0 & -15/2 & -15/2 \\ 0 & 4.5 & 9/2 \end{bmatrix} \begin{matrix} \sim \\ -\tfrac{2}{15}R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 1/2 & 5/2 \\ 0 & 1 & 1 \\ 0 & 4.5 & 9/2 \end{bmatrix} \begin{matrix} -\tfrac{1}{2}R_2+R_1 \to R_1 \\ \sim \\ -4.5R_2+R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 + 2x_3 = 0 & x_1 = -2x_3 \\ x_2 + x_3 = 0 & x_2 = -x_3 \end{matrix}$$

$$= \{(-2x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\}$$

$$= \text{span}\{(-2,-1,1)\} \qquad \dim(E_{-3}) = 1$$

(iv) find the set of all points in $\mathbb{R}^3$, say $Q = (a_1, a_2\ a_3)$, s.t. $A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 7.23 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

we dont need to make any calc. here

the thingy says that 7.23 is an eigen value but we found our eigen values to be 0, 3, -3

$$\therefore \quad A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 7.23 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \{(0,0,0)\}$$

Q6: let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & 4 \end{bmatrix}$

(i) find $C_A(\alpha)$

$$|\alpha I_n - A| = |\alpha I_3 - A|$$

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} \boxed{\alpha \quad 0 \quad 0} \\ -1 \quad \alpha \quad 4 \\ 0 \quad -1 \quad \alpha-4 \end{bmatrix}$$

$$= (-1)^2 (\alpha) \begin{vmatrix} \alpha & 4 \\ -1 & \alpha-4 \end{vmatrix}$$

$$= (\alpha) [\alpha(\alpha-4) - (-1)(4)]$$

$$= \alpha [\alpha^2 - 4\alpha + 4]$$

$$= \alpha^3 - 4\alpha^2 + 4\alpha$$

(ii) find all the eigen values of A

$$\alpha(\alpha^2 - 4\alpha + 4) = \alpha(\alpha-2)(\alpha-2)$$

$$\Rightarrow \alpha = 0 \quad \alpha = 2$$

(iii) for each eigen value of A, find the corresponding eigenspace and write it as span

$$\begin{bmatrix} \alpha & 0 & 0 \\ -1 & \alpha & 4 \\ 0 & -1 & \alpha-4 \end{bmatrix}$$

E_0 homog. system

$$= \begin{bmatrix} 0 & 0 & 0 \\ \underline{-1} & 0 & 4 \\ 0 & \underline{-1} & -4 \end{bmatrix} \qquad \begin{aligned} 0 &= 0 \\ -x_1 + 4x_3 &= 0 \\ -x_2 - 4x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= 4x_3 \\ x_2 &= -4x_3 \end{aligned}$$

reduced LOL

$$= \{ (4x_3, -4x_3, x_3) \mid x_3 \in \mathbb{R} \}$$

$$= \{ x_3 (4, -4, 1) \} = \text{span} \{ (4, -4, 1) \}$$

$$E_2 = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 4 \\ 0 & -1 & 2-4 \end{bmatrix} \overset{\frac{1}{2}R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 4 \\ 0 & -1 & -2 \end{bmatrix} \overset{R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{bmatrix} \overset{\frac{1}{2}R_2}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \overset{R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$

$x_2 + 2x_3 = 0 \quad x_2 = -2x_3$

$0 = 0$

$$= \{(0, -2x_3, x_3) \mid x_3 \in \mathbb{R}\}$$

$$= \text{span}\{(0,-2,1)\}$$

Q7: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x_1\ x_2\ x_3) =$

$$(x_1 + 2x_2 + 10x_3,$$
$$-x_1 + 4x_2 + 5x_3,$$
$$x_1 - 4x_2 - 5x_3)$$

(i) find $C_T(\alpha)$

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 1 & 2 & 10 \\ -1 & 4 & 5 \\ 1 & -4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha-1 & -2 & -10 \\ 1 & \alpha-4 & -5 \\ -1 & 4 & \alpha+5 \end{bmatrix} \overset{R_2+R_3 \to R_3}{\sim} \begin{bmatrix} \alpha-1 & -2 & -10 \\ 1 & \alpha-4 & -5 \\ \boxed{0 \quad \alpha \quad \alpha} \end{bmatrix}$$

$$= (-1)^5(\alpha) \begin{vmatrix} \alpha-1 & -10 \\ 1 & -5 \end{vmatrix} + (-1)^6(\alpha) \begin{vmatrix} \alpha-1 & -2 \\ 1 & \alpha-4 \end{vmatrix}$$

$$= -\alpha[(\alpha-1)(-5) + 10] + \alpha[(\alpha-1)(\alpha-4) + 2]$$

$$= -\alpha[-5\alpha + 5 + 10] + \alpha[\alpha^2 - 4\alpha - \alpha + 4 + 2]$$

$$= \cancel{5\alpha^2} - 15\alpha + \alpha^3 - \cancel{5\alpha^2} + 6\alpha$$

$$= \alpha^3 - 9\alpha$$

(ii) find all eigen values of T.

$$= \alpha(\alpha^2 - 9)$$

$$\alpha = 0, 3, -3$$

(iii) for each eigen value of T, find the corresponding eigenspace and write it as span

$\alpha = 0, 3, -3$

E_0 homog. system

$$\begin{bmatrix} \alpha-1 & -2 & -10 \\ 1 & \alpha-4 & -5 \\ -1 & 4 & \alpha+5 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & -10 \\ 1 & -4 & -5 \\ -1 & 4 & 5 \end{bmatrix} \overset{-1R_1}{\sim}$$

$$\begin{bmatrix} 1 & 2 & 10 \\ 1 & -4 & -5 \\ -1 & 4 & 5 \end{bmatrix} \overset{-R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 2 & 10 \\ 0 & -6 & -15 \\ -1 & 4 & 5 \end{bmatrix} \overset{-1/6 R_2}{\sim}$$

$$\begin{bmatrix} 1 & 2 & 10 \\ 0 & 1 & 5/2 \\ -1 & 4 & 5 \end{bmatrix} \underset{-4R_2+R_3 \to R_3}{\overset{-2R_2+R_1 \to R_1}{\sim}} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5/2 \\ -1 & 0 & -5 \end{bmatrix} \underset{-R_3}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5/2 \\ 1 & 0 & 5 \end{bmatrix} \overset{-R_3+R_1 \to R_1}{\sim} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 5/2 \\ 1 & 0 & 5 \end{bmatrix}$$

$x_2 + 5/2\, x_3 = 0 \qquad x_2 = -5/2\, x_3$

$x_1 + 5x_3 = 0 \qquad x_1 = -5x_3$

$= \{(-5x_3, -5/2\, x_3, x_3) \mid x_3 \in \mathbb{R}\}$

$= \text{span}\{(-5, -5/2, 1)\} \qquad \dim(E_0) = 1$

$$E_3 = \begin{bmatrix} 3-1 & -2 & -10 \\ 1 & 3-4 & -5 \\ -1 & 4 & 3+5 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & -10 \\ 1 & -1 & -5 \\ -1 & 4 & 8 \end{bmatrix} \overset{1/2 R_1}{\sim}$$

$$\begin{bmatrix} 1 & -1 & -5 \\ 1 & -1 & -5 \\ -1 & 4 & 8 \end{bmatrix} \underset{R_1+R_3 \to R_3}{\overset{-R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 1 & -1 & -5 \\ 0 & 0 & 0 \\ 0 & 3 & 3 \end{bmatrix} \underset{1/3 R_3}{\sim}$$

$$\begin{bmatrix} 1 & -1 & -5 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underset{R_3+R_1 \to R_1}{\sim} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$x_1 - 4x_3 = 0 \qquad x_1 = 4x_3$

$x_2 + x_3 = 0 \qquad x_2 = -x_3$

$= \{(4x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\}$

$= \text{span}\{(4, -1, 1)\} \qquad \dim(E_3) = 1$

$E_{-3}: \begin{bmatrix} -3-1 & -2 & -10 \\ 1 & -3-4 & -5 \\ -1 & 4 & -3+5 \end{bmatrix} \sim \begin{bmatrix} -4 & -2 & -10 \\ 1 & -7 & -5 \\ -1 & 4 & 2 \end{bmatrix} \begin{array}{c} -1/4 R_1 \\ \sim \end{array}$

$\begin{bmatrix} 1 & 1/2 & 5/2 \\ 1 & -7 & -5 \\ -1 & 4 & 2 \end{bmatrix} \begin{array}{c} -R_1 + R_2 \to R_2 \\ \sim \\ R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 1/2 & 5/2 \\ 0 & -15/2 & -15/2 \\ 0 & 4.5 & 9/2 \end{bmatrix} \begin{array}{c} \sim \\ -2/15 R_2 \end{array}$

$\begin{bmatrix} 1 & 1/2 & 5/2 \\ 0 & 1 & 1 \\ 0 & 4.5 & 9/2 \end{bmatrix} \begin{array}{c} -1/2 R_2 + R_1 \to R_1 \\ \sim \\ -4.5 R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 + 2x_3 = 0 \quad x_1 = -2x_3$

$\quad x_2 + x_3 = 0 \quad x_2 = -x_3$

$= \{ (-2x_3, -x_3, x_3) \mid x_3 \in \mathbb{R} \}$

$= \text{span} \{ (-2, -1, 1) \} \quad \dim(E_{-3}) = 1$

(iv) find the set of all points in $\mathbb{R}^3$, say $Q = (a_1\ a_2\ a_3)$, such that $T(a_1\ a_2\ a_3) = 7.2\,(a_1\ a_2\ a_3)$

its basically saying that 7.2 is an eigen value of T which is impossible thus $= \{(0,0,0)\}$

(v) is T one-to-one? is T onto?

1-1 IF $Z(T) = \{(0,0,0)\}$

but $\{(-5, -5/2, 1)\} \neq \{(0,0,0)\}$ so, its NOT 1-1

$\dim(Z(T)) + \dim(R(T)) = \dim(D)$

$1 + \dim(R(T)) = 3$

$\dim(R(T)) = 3 - 1 = 2 \neq \dim(\text{co-domain})$

$\therefore$ its onto

2^{nd} MD MATERIAL:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix}$$

Def: null $(A) \longrightarrow$ solution to the homogeneous system $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \approx \left[A \mid 0 \right]$

nullity $(A) = \dim(\text{Null}(A))$

Def: [ONLY FOR $n \times n$ matrices]

A, $n \times n$, we say A is non-singular (invertible) if $\exists$ a matrix denoted by $\underbrace{A^{-1} \text{ (the inverse of } A)}_{\neq \frac{1}{A}}$ s.t. $AA^{-1} = I_n$

exists

CAREFUL

Know: A, $n \times n$, is invertible IF $|A| \neq 0$

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

not invertible bcuz when calculating the $|A| = 0$

find the inverse (A^{-1}) if possible.

(1) $\left[A \mid I_n \right]$

(2) do the row op $\left[\text{if } I_n \mid A^{-1} \right]$

$\left[\text{if} \neq I_n \mid \underbrace{A^{-1} \text{ doesn't exist}} \right]$

$\therefore$ A is non-invertible / singular

$$\left[\begin{array}{cc|cc} 2 & 4 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{array}\right]$$

$R_1 \longleftrightarrow R_2$
$\sim$
$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 0 \end{array}\right]$$

$-2R_1 + R_2 \rightarrow R_2$
$\sim$
$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

B

$BA = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$\hookrightarrow$ no way that we get I_2 on this side so, its non-invertible

Result:

$$\left[\begin{array}{c|c} A_{n\times n} & B_{n\times n}\end{array}\right] \underset{\text{equivalent}}{\overset{\text{row op}}{\sim}} \left[\begin{array}{c|c} D & E\end{array}\right] \Rightarrow EA = D$$

always true :)
noice

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & -1 & 2 \\ 2 & 4 & -3 \end{bmatrix}$$ find A^{-1} if possible.

$$A = \left[\begin{array}{ccc|ccc} \underline{1} & 2 & -2 & 0 & 0 & 1 \\ -1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 4 & -3 & 1 & 0 & 0 \end{array}\right] \underset{-2R_1+R_3 \to R_3}{\overset{R_1+R_2 \to R_2}{\sim}} B = \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & \underline{1} & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\overset{-2R_2+R_1 \to R_1}{\sim} C = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & \underline{1} & -2 & 0 & 1 \end{array}\right] \overset{2R_3+R_1 \to R_1}{\sim} D = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\underbrace{}_{I_3} \qquad \underbrace{}_{A^{-1}}$$

$\therefore$ the matrix is invertible / non-singular

$$|A| = |B| = |C| = |I_3| = 1$$

Properties:

$$A A^{-1} = I_n$$

$$|A A^{-1}| = |I_n| = 1$$

$$|A|\,|A^{-1}| = 1 \,, \quad |A| \neq 0$$

Know: $$|A^{-1}| = \frac{1}{|A|}$$

$$A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 10 & 1 \\ -2 & 6 & 10 \end{bmatrix}$$ Solve the system. $$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

Know: $(A^{-1})^{-1} = A$

augmented matrix $$\left[\begin{array}{c|c} A & \begin{matrix} 2 \\ 4 \\ 7 \end{matrix} \end{array}\right]$$

$$A^{-1}\left[A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}\right] \approx \underbrace{A^{-1}A}_{I_3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 10 & 1 \\ -2 & 6 & 10 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 10 \\ 6 \end{bmatrix} + 7 \begin{bmatrix} 3 \\ 1 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} + \begin{bmatrix} 8 \\ 40 \\ 24 \end{bmatrix} + \begin{bmatrix} 21 \\ 7 \\ 70 \end{bmatrix} = \begin{bmatrix} 31 \\ 51 \\ 90 \end{bmatrix} \qquad \begin{matrix} x_1 = 31 \\ x_2 = 51 \\ x_3 = 90 \end{matrix}$$

solution set $= \{ (31, 51, 90) \}$ unique sol.

KNOW:

A, B are invertible $n \times n$. Then $(AB)^{-1} = B^{-1} A^{-1}$

ORDER MATTERS!

reminder:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Know: $C \to n \times m$ $D \to m \times n$

$$(CD)^T = D^T C^T$$

Special case: 2x2 ONLY

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ |A| \neq 0$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

ex: $A = \begin{bmatrix} 3 & 7 \\ 2 & 1 \end{bmatrix}$ $|A| = 3 - 14 = -11$

$$A^{-1} = \frac{-1}{11} \begin{bmatrix} 1 & -7 \\ -2 & 3 \end{bmatrix}$$

Know $\underset{n \times m}{A}$ $\underset{n \times m}{B}$

- $(A \pm B)^T = A^T \pm B^T$
- $(A^T)^T = A$

A, 2x2

$$= \left(\left(A \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \right)^T \right)^T = \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right)^T$$ Find A.

$$A \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right)^T$$

$$A \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$A \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}}_{B} = \begin{bmatrix} 0 & -3 \\ -3 & -4 \end{bmatrix}$$

calculate $B^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \frac{1}{1}$

$$\left(A \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}}_{B} = \begin{bmatrix} 0 & -3 \\ -3 & -4 \end{bmatrix} \right) B^{-1}$$

$$\underbrace{A \underbrace{B B^{-1}}_{}}_{A I_2 \to A} = \begin{bmatrix} 0 & -3 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$|A| \neq 0$ means A^{-1} exists

$$|A^{-1}| = \frac{1}{|A|}$$

A, $n \times n$, A^{-1} exists

Know:

- $(A^T)^{-1} = (A^{-1})^T$
- $|A^T| = |A|$

- A, nxn, assume A has at least two identical rows/cols then $|A| = 0$
  ↳ assume i^{th} row and k^{th} row are identical

$$\begin{array}{c} i: \\ k: \end{array}\begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix} \xrightarrow{-R_i + R_k \to R_k} \begin{bmatrix} \text{---} \\ 0 \cdots 0 \end{bmatrix} \quad \therefore |A| = |B|$$

$|A|$ $\qquad$ $|B|$

choose k^{th} row and find $|B|$. clearly $|B| = 0$ hence, $|A| = 0$

IF i^{th} col and k^{th} col. of A are identical, then i^{th} row & k^{th} row of A^T are identical since:

$$|A| = |A^T| \ \& \ |A^T| = 0 \ \therefore \text{ we have } |A| = 0$$

system of LE: assume nxn

aug matrix $\left[\begin{array}{c|c} A & \text{const} \end{array}\right]$ has a unique sol IF $|A| \neq 0$ & A^{-1} exist

coeff matrix

$\left[\begin{array}{c|c} A & \text{const} \end{array}\right]$ $|A| = 0$ → consistent : infinitley many sol.
→ inconsistent: has no sol.

A, 4x4

$$\begin{array}{ccc} \text{1st col} & & \text{4th col} \\ \end{array}\begin{bmatrix} | & - - - & | \\ | & - - - & | \\ | & - - - & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix} \text{1st col.}$$

1st col and 4th col are identical
sol: $(1,0,0,0)$
$(0,0,0,1)$ $\qquad |A| = 0$

$$x_1 \underset{\text{1st col}}{\begin{bmatrix} \\ \\ \end{bmatrix}} + x_2 \underset{\text{2nd col}}{\begin{bmatrix} \\ \\ \end{bmatrix}} + x_3 \underset{\text{3rd col}}{\begin{bmatrix} \\ \\ \end{bmatrix}} + x_4 \underset{\text{4th col}}{\begin{bmatrix} \\ \\ \end{bmatrix}} = \underset{\text{1st col}}{\begin{bmatrix} \\ \\ \end{bmatrix}}$$

$\underset{Q_1}{(1,2,3,4)}$ $\underset{Q_2}{(-1,4,6,8)}$ $\underset{Q_3}{(2,1,1,6)}$ $\underset{Q_4}{(0,0,1,2)}$ meaning of being dep / indep ?

$c_1Q_1 + c_2Q_2 + c_3Q_3 + c_4Q_4 = (0,0,0,0)$ $\quad c_1 = c_2 = c_3 = c_4 = 0$
Q_1, Q_2, Q_3, Q_4 are indep.

at least one of the c_i's $\neq 0$ → Q_1, Q_2, Q_3, Q_4 are dep.

$$A = \begin{array}{c} \begin{matrix} Q_1 & Q_2 & Q_3 & Q_4 \end{matrix} \\ \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 4 & 1 & 0 \\ 3 & 6 & 1 & 1 \\ 4 & 8 & 6 & 2 \end{bmatrix} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ has uniq sol. i.e. $(0,0,0,0)$ IF $|A| \neq 0$

Result:

Assume $Q_1, \dots, Q_n$ are points in $\mathbb{R}^n$ then $Q_1, \dots, Q_n$ are indep. IF

$$\left| \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} \right| \neq 0$$

cols or rows

Q: A, 4×4, $C_A(\alpha) = |\alpha I_4 - A| = (\alpha-3)^2(\alpha+5)^2$

note: it should be clear that A, $n\times n$, then $\deg(C_A(\alpha)) = n$

$\alpha = 3$ $\alpha = -5$ $\rightarrow$ repeated twice

$|A|$ = multiplication of the eigen values (w/ repetition)

$|A| = (3)(3)(-5)(-5) = (3)^2(-5)^2 = \#$

note:

if 0 is not an eigen value, then $|A| \neq 0$ $\therefore$ A^{-1} exists

α is an eigen value of A, $n\times n$, $|A| \neq 0$

$\exists$ nonzero point $(a_1, \dots, a_n)$

s.t. $A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \rightarrow A^{-1} A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha A^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$\rightarrow I_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha A^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \rightarrow \frac{1}{\alpha} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = A^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

so, $1/\alpha$ is an eigenvalue of A^{-1}

Q: A, 3×3, $C_A(\alpha) = (\alpha-2)^2(\alpha-4)$

(1) find $|A|$

(2) find $|A^{-1}|$ & find eigen values of A^{-1}

(3) given that $E_2 = \text{span}\{(1,0,2)\}$ $E_4 = \text{span}\{(0,2,3)\}$ find $E_{1/2}$ & $E_{1/4}$

(1) $\alpha = 2$ twice $\alpha = 4$ $|A| = (2)^2(4) = 16$

(2) $|A^{-1}| = 1/|A| = 1/16$ $\alpha = 1/2$ twice $\alpha = 1/4$

(3) $E_{1/2} = \left(A^{-1}[\ ] = \frac{1}{2}[\ ] \right)$ find all points in $\mathbb{R}^3$

$= E_2 = \text{span}\{(1,0,2)\}$

$E_{1/4} = E_4 = \text{span}\{(0,2,3)\}$

(4) $A^{-1}\begin{bmatrix}0\\6\\9\end{bmatrix} = \begin{bmatrix}0\\6/4\\9/4\end{bmatrix} = 1/4\begin{bmatrix}0\\6\\9\end{bmatrix}$

$A\begin{bmatrix}0\\6\\9\end{bmatrix} = 4\begin{bmatrix}0\\6\\9\end{bmatrix} = \begin{bmatrix}0\\24\\36\end{bmatrix}$

Trace (A), A must be nxn

$A = \begin{bmatrix}1 & 2 & 3\\-4 & 1 & 6\\2 & 2 & 10\end{bmatrix}$ trace (A) = add the #'s on the main diagonal

$= 1 + 1 + 10 = 12$

Def: A, nxn:

Trace |A| = sum of the numbers on the main diagonal

Result (know):

Trace |A| = sum of the eigen values (w/ repitition)

$C_A(\alpha) = (\alpha+1)^2(\alpha-3)^3(\alpha+4)$, note A is 6x6

find all eigen values of A by staring

$\alpha = -1$ (repeated twice)

$\alpha = -4$

$\alpha = -3$ (repeated three times)

trace $|A| = -1 + -1 + -4 + 3 + 3 + 3 = 3$

$|A| = (-1)(-1)(-4)(3)(3)(3) = -108$

find all eigenvalues of A^{-1}

(-1) repeated twice

$(-1/4)$

$(1/3)$ repeated three times

$E_{1/3}$ (w.r.t. A^{-1}) $= E_3$ (w.r.t. to A)

$Q \in E_3$, $Q \neq (0,0,\cdots,0)$

$AQ^T = 3Q^T$ $\quad A^{-1}Q^T = 1/3\, QT$

Q: α is an eigenvalue of A (A is nxn) $\exists\ Q \in E_\alpha$ [$Q \neq (0,0,0,0)$,
$Q = (a_1, \dots, a_n)$

$AQT = \alpha QT$

multiply by A

$A^2 Q^T = \alpha AQT$

$A^2 Q^T = \alpha \alpha QT = \alpha^2 QT$ α is an eigenvalue of A

NOTE:

if α is an eigenvalue of A, then α^K is an eigenvalue of A^K

$\Rightarrow E_\alpha$ (w.r.t. A) $= E_{\alpha^K}$ (w.r.t. A^K)

Q: A, 3x3

$C_A(\alpha) = |\alpha I_3 - A| = (\alpha-4)^2(\alpha+4)$

$B = 2A^2 + 5A^{-1} - 4I_3$

find det (B) and Trace (B)

Know:

$|B| \neq |2A^2| + |5A^{-1}| + |-4I_3|$

eigenvalues of A are:

4 repeated twice, -4

for $\alpha = 4$: $2(4)^2 + 5(1/4) - 4 = 29.25$

this is the eigenvalue of B

for $\alpha = -4$: $2(-4)^2 + 5(-1/4) - 4 = 26.75$

Trace (B) = 29.25 + 29.25 + 26.75

For understanding:

$Q \in E_4 \quad Q \neq (0,0,0)$

$AQ^T = 4Q^T$

$BQ^T = [2A^2 + 5A^{-1} - 4I_3]\, Q^T$

$= 2A^2Q^T + 5A^{-1}Q^T - 4I_3Q^T$

$= 32\,Q^T + 5\,(\underset{\text{or } -4}{1/4})\,Q^T - 4Q^T = (32 + 5/4 - 4)\,Q^T$

Know :

α is an eigenvalue of A

(1) $\alpha^{-1} = \frac{1}{\alpha}$ $(\alpha \neq 0)$ is an eigenvalue of A^{-1}

(2) α^{K} is an eigenvalue of A^{K}

(3) c is a constant , $C\alpha$ is an eigenvalue of CA

Def: A, nxn , we say A is diagnolizable , if $\exists$ an invertible matrix Q , and a diagonal matrix D s.t.

$$\underbrace{Q^{-1} A Q = D} \xrightarrow{\text{solve for } A} \underbrace{A = Q D Q^{-1}}$$

same, just solving for D or A

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} 1/1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \rightarrow A^{4} = \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 2^4 \end{bmatrix}$$

so $A^{n} = \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 2^n \end{bmatrix}$

calculate A^2 by equation

$$A^2 = (QDQ^{-1})(QDA^{-1})$$
$$= Q D^2 Q^{-1}$$

$$A^3 = Q D^3 Q^{-1}$$

so $A^n = Q D^n Q^{-1}$

Cramer can be used when solving system of LE , nxn ,

$$|\text{coeff matrix}| \neq 0$$

Q: solve for X_2 only

$$X_1 + 2X_2 - X_3 = 0$$
$$-X_1 + 5X_2 + 2X_3 = 2$$
$$2X_1 + 4X_2 + 10X_3 = 10$$

$$|\text{coeff matrix}| = |A| = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 5 & 2 \\ 2 & 4 & 10 \end{vmatrix}$$

$$x_2 = \frac{|[x_1 \; c \; x_3]|}{|A|} = \frac{\begin{array}{ccc} x_1 & c & x_3 \\ \end{array} \\ \begin{vmatrix} 1 & 0 & -1 \\ -1 & 0 & 2 \\ 2 & 10 & 10 \end{vmatrix}}{|A|}$$

$$x_3 = \frac{|[x_1 \; x_2 \; c]|}{|A|} = \frac{\begin{array}{ccc} x_1 & x_2 & c \\ \end{array} \\ \begin{vmatrix} 1 & 2 & 0 \\ -1 & 5 & 2 \\ 2 & 4 & 10 \end{vmatrix}}{|A|}$$

Adjoint method:

given

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad n \times n$$

$a_{3,4}$ = 3rd row 4th col

$$\text{adjoint of } A = C = \underbrace{\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{12} \\ & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}}_{n \times n}$$

(i,k) - entry of C $= C_{i,k}$ ← row (i), ← col (k)

$$= \frac{(-1)^{i+k} \begin{vmatrix} A \text{ after deleting} \\ \underbrace{k^{th} \text{ row} \;\; i^{th} \text{ col}}_{\text{you switch them}} \text{ of } A \end{vmatrix}}{|A|}$$

know:

$$\boxed{A \cdot \text{adjoint}(A) = \det(A) \cdot I_n}$$

Assume $\det(A) \neq 0$

$\Rightarrow A^{-1}$ exists

$$A \left[\underbrace{\frac{1}{|A|} \text{adjoint}(A)}_{A^{-1}} \right] = I_n$$

$$A A^{-1} = I_n$$

Q:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -2 & 6 & 1 \\ -2 & -3 & 5 \end{bmatrix}$$ find the (2-3)-entry of A^{-1}

one way to do it is to find the inverse (A^{-1}) and stare and you'll find $A^{-1}_{2,3}$

OR

$$= \frac{(-1)^{i+k} \left| A \text{ after deleting } A_{3,2} \right|}{|A|}$$

change to triangular

$$\begin{bmatrix} 2 & 3 & 4 \\ -2 & 6 & 1 \\ -2 & -3 & 5 \end{bmatrix} \underset{R_1+R_3 \to R_3}{\overset{R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 9 & 5 \\ 0 & 0 & 9 \end{bmatrix}$$

$$|A| = (2)(9)(9) = 162$$

$$A_{2,3} = \frac{(-1)^5 \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix}}{162} = \frac{-1\ (2+8)}{162} = {-10}/{162} = -5/81$$

so, when you find A^{-1} immediatley then you should find this number $\to A_{2,3}$

Know (Result):

(1) Assume $C_A(\alpha) = (\alpha - a_1)^{n_1} (\alpha - a_2)^{n_2} \cdots (\alpha - a_k)^{n_k}$

$0 < \dim(E_{a_i}) \leq n_i$

(2) A, $n \times n$, diagnolizable IF $\forall$ eigenvalue a_i,

for all

$\dim(E_{a_i}) = n_i$

Note:

A, $n \times n$, is diagnolizable IF $\exists$ a diagonal matrix & invertible matrix Q s.t.

$$Q^{-1} A Q = D \iff A = Q D Q^{-1}$$

$\underline{Q}$: A, 3×3

$$C_A(\alpha) = (\alpha-2)^{\overset{n_2}{2}}(\alpha+4)$$

E_2 = span $\{(1,3,2)\}$ $\rightarrow$ $\dim(E_2) = 1$

E_{-4} = span $\{(0,1,5)\}$ $\rightarrow$ $\dim(E_{-4}) = 1$

$$A\begin{bmatrix}-2\\-6\\-4\end{bmatrix} = 2\begin{bmatrix}-2\\-6\\-4\end{bmatrix}$$

is A diagnolizable ?

no, bcuz $\dim(E_2) \neq n_2$ $[n_2 = 2]$

$\underline{Q}$: A, 5×5

$$C_A(\alpha) = (\alpha-3)^{\overset{n_3}{2}}(\alpha+5)^{\overset{n_{-5}}{2}}(\alpha-6)^{\overset{n_6}{1}}$$

E_3 = span $\{(1,1,1,1,1)(-1,1,1,1,1)\}$ $\dim(E_3) = 2 = n_3$

E_{-5} = span $\{(-1,-1,1,1,1)(-1,-1,-1,1,1)\}$ $\dim(E_{-5}) = 2 = n_{-5}$

E_6 = span $\{(0,0,0,0,1)\}$ $\dim(E_6) = 1 = n_6$

$\therefore$ A is diagnolizable

find a diagonal matrix, D, and an invertible matrix, Q, s.t. $Q^{-1}AQ = D$

$$D = \begin{bmatrix}3&0&0&0&0\\0&-5&0&0&0\\0&0&-5&0&0\\0&0&0&6&0\\0&0&0&0&3\end{bmatrix} \xrightarrow{\text{corresponds to}} Q = \begin{bmatrix}1&-1&-1&0&-1\\1&-1&-1&0&1\\1&1&-1&0&1\\1&1&1&0&1\\1&1&1&1&1\end{bmatrix}$$

$\underline{\underline{OR}}$

$$D = \begin{bmatrix}6&0&0&0&0\\0&3&0&0&0\\0&0&-5&0&0\\0&0&0&3&0\\0&0&0&0&-5\end{bmatrix} \longrightarrow Q = \begin{bmatrix}0&1&-1&-1&-1\\0&1&-1&1&-1\\0&1&1&1&-1\\0&1&1&1&1\\ \underbrace{1}_{A_6}&1&1&1&1\end{bmatrix}$$

$$Q^{-1} A Q = D$$

$$\underbrace{\begin{bmatrix} \text{———} \\ \\ \\ \end{bmatrix}}_{Q^{-1}} \underbrace{\begin{bmatrix} 5 \times 5 \\ \\ \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{A_6} = 6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \underbrace{Q^{-1}}_{\text{whole matrix}} \underbrace{Q}_{A_6} : \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot 6 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>Q:</u> $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$ IF A is diagnolizable, find a diagonal matrix, D, and invertible matrix, Q, s.t. $Q^{-1} A Q = D$

$$C_\alpha(A) = |\alpha I_3 - A|$$

$$= \begin{bmatrix} \alpha - 2 & 0 & 0 \\ 0 & \alpha - 2 & 0 \\ 1 & -1 & \alpha - 3 \end{bmatrix} = (\alpha - 2)^2 (\alpha - 3)$$

$\alpha = 2$ repeated twic

$\alpha = 3$

E_2 = Sol. set of the homog sys. $[2 I_3 - A \mid 0]$

$$= \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & c \end{matrix} \\ \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \end{array}$$

$x_1 - x_2 - x_3 = 0$ $\qquad$ $x_1 \rightarrow$ leading var

$x_1 = x_2 + x_3$ $\qquad$ $x_2 \; x_3 \rightarrow$ free var

$$= \{ (x_2 + x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R} \}$$

$$E_2 = \text{span} \{ (1,1,0) \; (1,0,1) \}$$

$$\dim(E_2) = 2$$

$$E_3 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \underset{R_2 + R_3 \rightarrow R_3}{\overset{-R_1 + R_3 \rightarrow R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ 0 = 0 \end{matrix}$$

leading var: $x_1 \; x_2$

free var: x_3

$$E_3 = \{ (0, 0, x_3) \mid x_3 \in \mathbb{R} \} = \text{span} \{ (0,0,1) \}$$

$\dim(E_3) = 1$ $\qquad$ $\therefore$ A is diagnolizable

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Worksheet:

Q1: Use Cramers Rule and solve for X_3

$$X_1 + 2X_2 - X_3 = 10$$
$$-2X_1 + 4X_2 + 2X_3 = -6$$
$$-X_1 - 2X_2 + 6X_3 = 4$$

$$A = \begin{array}{cccc} X_1 & X_2 & X_3 & C \\ \end{array}$$
$$A = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 10 \\ -2 & 4 & 2 & -6 \\ -1 & -2 & 6 & 4 \end{array}\right] \begin{array}{c} 2R_1 + R_2 \to R_2 \\ \sim \\ R_1 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 10 \\ 0 & 8 & 2 & 14 \\ 0 & 0 & 5 & 14 \end{array}\right]$$

$$|A| = (1)(8)(5) = 40$$

$$X_3 = \frac{\begin{bmatrix} 1 & 2 & 10 \\ -2 & 4 & -6 \\ -1 & -2 & 4 \end{bmatrix}}{|A|} \begin{array}{c} 2R_1 + R_2 \to R_2 \\ \sim \\ R_1 + R_3 \to R_3 \\ = \end{array} \frac{\begin{bmatrix} 1 & 2 & 10 \\ 0 & 8 & 14 \\ 0 & 0 & 14 \end{bmatrix}}{40} = \frac{(1)(8)(14)}{40} = \frac{112}{40}$$

Q2: Use Cramers Rule and solve for X_2

$$X_1 + 2X_2 - X_3 + X_4 = -1$$
$$-2X_1 + 4X_2 + 2X_3 + 5X_4 = -8$$
$$-X_1 - 2X_2 + 6X_3 + X_4 = 1$$
$$3X_1 + 6X_2 - 3X_3 + 6X_4 = -3$$

$$A = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ -2 & 4 & 2 & 5 & -8 \\ -1 & -2 & 6 & 1 & 1 \\ 3 & 6 & -3 & 6 & -3 \end{array}\right] \begin{array}{c} 2R_1 + R_2 \to R_2 \\ \sim \\ R_1 + R_3 \to R_3 \\ -3R_1 + R_4 \to R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ 0 & 8 & 0 & 7 & -6 \\ 0 & 4 & 5 & 2 & 2 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right] \begin{array}{c} -\tfrac{1}{2}R_2 + R_3 \to R_3 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ 0 & 8 & 0 & 7 & -6 \\ 0 & 0 & 5 & -3/2 & 5 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right]$$

$$|A| = (1)(8)(5)(3) = 120$$

$$\begin{bmatrix} \underline{\underline{1}} & -1 & -1 & 1 \\ -2 & -8 & 2 & 5 \\ -1 & 1 & 6 & 1 \\ 3 & -3 & -3 & 6 \end{bmatrix} \underset{\substack{R_1+R_3\to R_3 \\ -3R_1+R_4\to R_4}}{\overset{2R_1+R_2\to R_2}{\sim}} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & -10 & 0 & 7 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$|\, x_1 \;\; c \;\; x_3 \;\; x_4 \,| = -150$$

$$\therefore\; x_2 = \frac{-150}{120}$$

Q3: let $A \begin{bmatrix} 2 & 4 \\ -2 & 3 \end{bmatrix}$ if A^{-1} exists, then find A^{-1}

A^{-1} exists if $|A| \neq 0$

$$|A| = (2)(3) - (4)(-2) = 14 \neq 0 \quad \therefore\; A^{-1} \text{ exists}$$

$$\left[\, A_2 \;\middle|\; I_2 \,\right] \sim \left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right]$$

OR 2x2 hack !

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{14} \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3/14 & -4/14 \\ 2/14 & 2/14 \end{bmatrix}$$

Q4: let $A = \begin{bmatrix} 2 & 11 \\ 1 & 5 \end{bmatrix}$ find A^{-1} if possible

$$|A| = (2)(5) - (1)(11) = -1 \neq 0$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -11 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 11 \\ 1 & -2 \end{bmatrix}$$

Q5: find the matrix 2x3 such that

$$\left(\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} A + \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \right)^T = \begin{bmatrix} -1 & 3 \\ 0 & 4 \\ -1 & 0 \end{bmatrix} \right)^T$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} A + \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 3 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} A = \begin{bmatrix} -1 & 0 & -1 \\ 3 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

MUST MULT. FROM THE LEFT

$$\underline{\underline{A^{-1}}} \left(\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}}_{A^{-1} = ?} A = \begin{bmatrix} -4 & -2 & -2 \\ 3 & 3 & 1 \end{bmatrix} \right)$$

$$|A| = (1)(1) - (2)(1)$$

$$A^{-1} = -1 \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$AA^{-1} = AI_2 = A = \begin{bmatrix} \boxed{-1 \quad 2} \\ 1 \quad -1 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix}-4 \\ 3\end{matrix}} & \begin{matrix}-2 \\ 3\end{matrix} & \begin{matrix}-2 \\ 1\end{matrix} \end{bmatrix}$$

dot product = $(-1)(-4) + (2)(3) = -10$

$$= \begin{bmatrix} 10 & 8 & 4 \\ -7 & -5 & -3 \end{bmatrix}$$

<u>Q6:</u> find the matrix A, 3×4 such that

$$\left(A^T \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \right)^T$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & \underline{\underline{1}} & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} -2R_2 + R_3 \to R_3 \\ \sim \\ R_2 + R_1 \to R_1 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$\underbrace{}_{I_3} \qquad \underbrace{}_{A^{-1}}$$

$$AA^{-1} = I_3 A = A = \begin{bmatrix} \boxed{1 \quad 1 \quad 0} \\ 0 \quad 1 \quad 0 \\ 0 \quad -2 \quad 1 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix}1 \\ 0 \\ 1\end{matrix}} & \begin{matrix}0 \\ 1 \\ 1\end{matrix} & \begin{matrix}1 \\ 1 \\ 1\end{matrix} & \begin{matrix}1 \\ 1 \\ 0\end{matrix} \end{bmatrix}$$

3×3 $\qquad$ 3×4

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & -1 & -2 \end{bmatrix}$$

$$\left(A^T \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \right) \underline{\underline{A^{-1}}}$$

FROM THE RIGHT SIDE

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^T I_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \right)^T \sim A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & -1 & -2 \end{bmatrix}$$

Q7: let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ where $a_1 \cdots c_3$ are some real numbers such that $|A| \neq 0$. find the solution set of the system of linear equations

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3a_1 - 5a_3 \\ 3b_1 - 5b_3 \\ 3c_1 - 5c_3 \end{bmatrix}$$

$$x_1 (1^{st} \text{ col of } A) + x_2 (2^{nd} \text{ col of } A) + x_3 (3^{rd} \text{ col of } A) = \begin{bmatrix} 3a_1 - 5a_3 \\ 3b_1 - 5b_3 \\ 3c_1 - 5c_3 \end{bmatrix}$$

by staring, you can see:

$x_1 = 3 \quad x_2 = 0 \quad x_3 = -5$

$|A| \neq 0 \quad \therefore$ unique sol

$= \{ (3, 0, -5) \}$

Q8: given A is a 4×4 matrix and $C_A(\alpha) = |\alpha I_4 - A| = (\alpha - 2)^2 (\alpha + 2)(\alpha - 3)$

let $B = 2A^4 + 3A^2 - 2I_4$. Find B & Trace (B).

$\alpha = 2$ repeated twice

$\alpha = -2$

$\alpha = 3$

$|A| = (2)(2)(-2)(3) = -24$

eigen values of B:

$\alpha = 2 \qquad 2(2^4) + 3(2^2) - (2) = 42$

$\alpha = 2 \qquad 2(2)^4 + 3(2)^2 - (2) = 42$

$\alpha = -2 \qquad 2(-2)^4 + 3(-2)^2 - (2) = 42$

$\alpha = 3 \qquad 2(3)^4 + 3(3)^2 - (2) = 187$

$|B| = (42)(42)(42)(187)$

Trace $(B) = 42 + 42 + 42 + 187$

Q9: given A is a 3×3 matrix and $C_A(\alpha) = |\alpha I_3 - A| = (\alpha-2)^2(\alpha+2)$
let $B = A^2 + 4A^{-1} + 3I_3$. Find $|B|$ & Trace (B)

$\alpha = 2$ repeated twice
$\alpha = -2$

find eigenvalues of B

$\alpha = 2 \rightarrow (2)^2 + 4(1/2) + 3 = 9$
$\alpha = 2 \rightarrow (2)^2 + 4(1/2) + 3 = 9$
$\alpha = -2 \rightarrow (-2)^2 + 4(-1/2) + 3 = 5$

$|B| = (9)(9)(5) = 405$

Trace $(B) = 9+9+5 = 23$

Q10: let A and B in Q9. Assume that $E_2 = \text{span}\{(1,0,1)(0,1,2)\}$
and $E_{-2} = \text{span}\{(-1,-1,1)\}$

let $Q = 3(1,0,1) + -2(0,1,2) = (3,-2,-1)$ then $Q \in E_2$
let $F = 5(-1,-1,1) = (-5,-5,5)$ then $F \in E_{-2}$

(i) find $A^{-1}\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$

$(3,-2,-1) \in Q \in E_2$

$$A^{-1}\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = 1/2\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -1 \\ -0.5 \end{bmatrix}$$

(ii) find $B\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$

Q9

A B

$(3,-2,-1) \in E_2 \in E_9$

$$\therefore \quad B\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = 9\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 27 \\ -18 \\ -9 \end{bmatrix}$$

(iii) find $B\begin{bmatrix} -5 \\ -5 \\ 5 \end{bmatrix}$

$(5,5,-5) \in E_{-2} \in E_5$

$$= 5\begin{bmatrix} -5 \\ -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 25 \\ 25 \\ -25 \end{bmatrix}$$

Q11: let A, B as in Q8. Assume

E_2 (w.r.t A) = span$\{(1,2,3,0)(-1,-2,-3,1)\}$

E_{-2} (w.r.t. A) = span$\{(-1,-2,2,0)\}$

find E_{42} (w.r.t. B)

E_{42} corresponds to $\alpha = 2$ repeated twice
$\alpha = -2$

$\therefore$ E_{42} (w.r.t. B) = $E_2 + E_{-2}$

= span$\{(1,2,3,0)(-1,-2,-3,1)(-1,-2,2,0)\}$

Q12: let $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ if possible find A^{-1}. then find $|A|$

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\xleftrightarrow{R_1 \quad R_2}}$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 + R_3 \to R_3 \\ \sim \\ -R_1 + R_4 \to R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} R_2 + R_1 \to R_1 \\ \sim \\ 2R_2 + R_3 \to R_3 \\ -2R_2 + R_4 \to R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right] \underset{\frac{1}{2} R_3}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right] \underset{-R_3 + R_2 \to R_2}{\sim}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right]$$

you did something wrong but still got the right answ so we vibin :)

$\neq I_4$ $\therefore$ A^{-1} doesn't exist

$|A| = 0$

Q13: given A, B are 2x2 matrix s.t.

$$A^{-1} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 0 & 4 \\ 2 & 5 \end{bmatrix}$$

(i) find $(AB)^{-1}$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$= \begin{bmatrix} 0 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 12 \\ 4 & 19 \end{bmatrix}$$

(ii) find $|3A^{-1}B|$

2x2 so,

$$3^n |A^{-1}B| = 3^2 |A^{-1}||B|$$

$$= 9(6)(1/-8) = -27/4$$

$$|A^{-1}| = (2)(3) - (2)(0) = 6$$

$$|B^{-1}| = (0)(5) - (4)(2) = -8$$

(iii) find $3|A^{-1}B|$

$$= 3|A^{-1}|\left(\frac{1}{|B^{-1}|}\right)$$

$$= 3(6)(-1/8) = -9/4$$

Q14: let $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 2 & -1 \\ 2 & -2 & -2 & 1 \end{bmatrix}$

(i) if possible, find A^{-1}

$$\left[\begin{array}{cccc|cccc} 0 & \underline{\underline{1}} & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 2 & -2 & -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_2 \to R_2 \\ \sim \\ -R_1 + R_3 \to R_3 \\ 2R_1 + R_4 \to R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 + R_3 \to R_3 \\ \sim \\ -2R_2 + R_4 \to R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} -R_3 + R_1 \to R_1 \\ \sim \end{array}$$

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 1 & 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} -R_4 + R_1 \to R_1 \\ \sim \\ R_4 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right] \quad R_1 \longleftrightarrow R_2$$

$$\left[\underbrace{\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}}_{I_4} \middle| \underbrace{\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & 0 & 1 \end{array}}_{A^{-1}}\right] \quad \therefore A^{-1} \text{ exist}$$

(ii) find the solution set to the system of LE.

$$(*) \quad A^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

3 methods: aug matrix, cramer, or A^{-1}

since we already have A^{-1} we can

$$(A^{-1})^T = (A^T)^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -2 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

$$I_4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underset{4 \times 4}{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -2 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}} \underset{4 \times 1}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore = \{(1, -2, 1, 2)\}$

Result.

$T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

T is invertible IF $n = m$ & T is $\underbrace{\text{isomorphism}}_{\text{1-1 \& onto}}$

dim (domain) = dim (range) + dim (zero)

$n = m + 0$

$T: \mathbb{R}^3 \to \mathbb{R}^3$ ← identity linear transformation

$T(a_1\ a_2\ a_3) = (a_1\ a_2\ a_3)$

$T(1, 0, 3) = (1, 0, 3) \ldots$

Standard matrix presentation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

remember:

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(a_1\ a_2) = (a_1 + 2a_2, -a_1 + a_2) \rightleftarrows T(a_1\ a_2) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

T is a LT bcuz its a linear comb

is T invertible ?

step ① : find the standard matrix presentation of T, M

$M = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ $\quad$ T is invertible IF M^{-1} exists

$|M| = (1)(1) - (2)(-1) = 3 \neq 0 \therefore$ m is invertible

& T is invertible

if T is invertible, find T^{-1}

$T^{-1}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T^{-1}(a_1, a_2) = M^{-1}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$M^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix}$

$\therefore \; T^{-1}(a_1, a_2) = \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1/3\, a_1 - 2/3\, a_2 \\ 1/3\, a_1 + 1/3\, a_2 \end{bmatrix}$

$T^{-1}(a_1, a_2) = (1/3\, a_1 - 2/3\, a_2 \,,\; 1/3\, a_1 + 1/3\, a_2)$

note:

$T \underbrace{\circ}_{\text{composition}} T^{-1} = \underbrace{I}_{\text{identity func.}}$

$(T \circ T^{-1})(a_1, a_2) = (a_1, a_2)$

fact:

$T_1: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$T_2: \mathbb{R}^k \rightarrow \mathbb{R}^n$

M_1 = standard matrix for T_1

↳ dim (co-domain) x dim (domain)

$m \times n$

M_2 = standard matrix for T_2

↳ $n \times k$

find the standard matrix presentation of $T_1 \circ T_2$

Answer: $M = \underset{m \times n}{M_1}\; \underset{n \times k}{M_2}$ $\qquad T_1 \circ T_2 : \mathbb{R}^k \longrightarrow \mathbb{R}^m$

$T_1: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T_1(a_1, a_2) = (a_1 + a_2 \,,\, -a_1)$

$T_2: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(a_1, a_2) = (3a_1 - a_2 \,,\; a_1 + a_2)$

Find $T_1 \circ T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T_1 \circ T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$= \begin{bmatrix} 4a_1 + 0 \\ -3a_1 + a_2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$f(x) = 3x^2 - 6x + 7$

find $f(A)$

$= \underbrace{3A^2}_{} - \underbrace{6A}_{} \boxed{+7}$ → PROBLEM!

you only add / sub matrices of the same size

matrix is 2x2

$= 3A^2 - 6A + 7I_2$ → no longer a problem

note:

if A, $n \times m$, $n \neq m$

$A^3 = A \; A \; A$ → is undefined

$n \times m \quad n \times m \quad n \times m$

cant multiply

$\therefore A^k$ is undefined

A, $n \times n$,

A is invertible

$A^{-5} = (A^{-1})^5 \qquad \therefore A^{-n} = (A^{-1})^n$

$A^{1/2}$ → undefined

Q: $A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$

$C_A(\alpha) = |\alpha I_2 - A|$

$= \begin{bmatrix} \alpha & -2 \\ 0 & \alpha - 1 \end{bmatrix} = \alpha(\alpha - 1)$

$\alpha = 1 \quad \alpha = 0$

instead of 1

$C_A(A) = A(A - I_2)$

$= \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Caley's Theorem:

A, $n \times n$,

$C_A(A) = \underbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}}_{n \times n}$

$C_A(\alpha) = \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1 \alpha + a_0$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \qquad C_A(\alpha) = ?$$

$$= |\alpha I_3 - A|$$

$$= \begin{vmatrix} \alpha-1 & 0 & -2 \\ 0 & \alpha-2 & -3 \\ 0 & 0 & \alpha-4 \end{vmatrix} = (\alpha-1)(\alpha-2)(\alpha-4)$$

$C_A(A) = ?$

$$= (A - I_3)(A - 2I_3)(A - 4I_3)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>Q:</u> find (1,3)-entry of A^{-1}

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 2 & 5 \\ -1 & -2 & 6 \end{bmatrix}$$

$$A \underset{R_1+R_3 \to R_3}{\overset{R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 9 \\ 0 & 0 & 10 \end{bmatrix}$$

$$|A| = |B| = 40$$

$$= \frac{(-1)^{1+3} \begin{vmatrix} \text{deleting} \\ \text{row } 3 \\ \text{col } 1 \end{vmatrix}}{|A|} = \frac{2}{40} = \frac{1}{20}$$

worksheet:

Q: let $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}_{2\times 2}$. is A diagnolizable?

$$C_A(\alpha) = |\alpha I_2 - A|$$

$$= \begin{vmatrix} \alpha-2 & -3 \\ 0 & \alpha-2 \end{vmatrix} = (\alpha-2)^{\underline{2}} \leftarrow n_i$$

$\alpha = 2$ twice

$$\begin{array}{cc} & \begin{matrix} x_1 & x_2 \end{matrix} \\ E_2 : & \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix} \end{array} = \begin{matrix} -3x_2 = 0 \\ x_2 = 0 \end{matrix}$$

leading x_2
free x_1

$$E_2 : \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$$
$$= \text{span}\{(1,0)\}$$

note:

A, $n\times n$, is diagnolizable IF $\forall$ eigenvalue a_i, $\dim(a_i) = n_i$

in this case, its 2

$\dim(E_2) = 1 \neq 2$

$\therefore$ its not diagnolizable

Q2: let $A = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is A diagnolizable? if yes, then find the diagonal matrix D and an invertible matrix Q such that $Q^{-1}AQ = D$

$$C_A(\alpha) = |\alpha I_4 - A|$$

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ 1 & \alpha & 0 & 0 & 4 \\ 2 & -1 & \alpha & 0 & 0 \\ 3 & 0 & -1 & \alpha & -5 \\ 4 & 0 & 0 & -1 & \alpha \end{array} = (-1)^2(\alpha) \begin{array}{c|ccc|} & 1 & 2 & 3 \\ 1 & \alpha & 0 & 0 \\ 2 & -1 & \alpha & -5 \\ 3 & 0 & -1 & \alpha \end{array} + (-1)^{4+1}(4) \begin{array}{c|ccc|} & 1 & 2 & 3 \\ 1 & -1 & \alpha & 0 \\ 2 & 0 & -1 & \alpha \\ 3 & 0 & 0 & -1 \end{array}$$

$$= (\alpha)\,(\cancel{-1})^{\cancel{2}\,1}(\alpha) \underbrace{\begin{vmatrix} \alpha & -5 \\ -1 & \alpha \end{vmatrix}}_{ad-bc} + (-4)(-1)(\cancel{-1})^{\cancel{6}\,1} \begin{vmatrix} -1 & \alpha \\ 0 & -1 \end{vmatrix}$$

$$= \alpha^2 [(\alpha)(\alpha) - (-5)(-1)] + 4[(-1)(-1) - (\alpha)(0)]$$

$$= \alpha^2 [\alpha^2 - 5] + 4$$

$$= \alpha^4 - 5\alpha^2 + 4 = (\alpha^2 - 1)(\alpha^2 - 4) = 0$$

$$\alpha = 1, -1, 2, -2$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 4 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -5 \\ 0 & 0 & -1 & 1 \end{bmatrix} \underset{\sim}{R_1+R_2 \to R_2} \left|\begin{matrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & -1 & 1 & -5 \\ 0 & 0 & -1 & 1 \end{matrix}\right| \underset{\sim}{R_2+R_3 \to R_3} \left|\begin{matrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{matrix}\right|$$

$$\underset{\sim}{R_3+R_4 \to R_4} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \left|\begin{matrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{matrix}\right| \end{array}$$

$x_1 + 4x_4 = 0$

$x_2 + 4x_4 = 0$

$x_3 - x_4 = 0$

$0 = 0$

x_1, x_2, x_3 leading

x_4 free

$= \{ (-4x_4, -4x_4, x_4, x_4) \mid x_4 \in \mathbb{R} \}$

$E_1 = \text{span} \{ (-4, -4, 1, 1) \}$

$$E_2 = \left|\begin{matrix} 2 & 0 & 0 & 4 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & -5 \\ 0 & 0 & -1 & 2 \end{matrix}\right| \underset{\sim}{\tfrac{1}{2}R_1+R_2 \to R_2} \left|\begin{matrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 2 \\ 0 & -1 & 2 & -5 \\ 0 & 0 & -1 & 2 \end{matrix}\right| \underset{\sim}{\tfrac{1}{2}R_2+R_3 \to R_3} \left|\begin{matrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & -1 & 2 \end{matrix}\right|$$

$$\underset{\sim}{\tfrac{1}{2}R_3+R_4 \to R_4} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \left|\begin{matrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{matrix}\right| \end{array}$$

$2x_1 + 4x_4 = 0$

$2x_2 + 2x_4 = 0$

$2x_3 - 4x_4 = 0$

$0 = 0$

$x_1 = -2x_4$

$x_2 = -x_4$

$x_3 = 2x_4$

$= \{ (-2x_4, -x_4, 2x_4, x_4) \mid x_4 \in \mathbb{R} \}$

$= \{ (-2, -1, 2, 1) \}$

$$E_{-1} = \left|\begin{matrix} -1 & 0 & 0 & 4 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & -1 & -1 \end{matrix}\right| \quad \ldots\ldots \text{ etc.}$$

$E_{-1} = \text{span} \{ (4, -4, -1, 1) \}$

$E_{-2} = \text{span} \{ (2, -1, -2, 1) \}$

$$Q = \left| \begin{matrix} E_2 & E_1 & E_{-1} & E_{-2} \end{matrix} \right| = \left|\begin{matrix} -2 & -4 & 4 & 2 \\ -1 & -4 & -4 & -1 \\ 2 & 1 & -1 & -2 \\ 1 & 1 & 1 & 1 \end{matrix}\right|$$

why this order?

$$D = \left|\begin{matrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{matrix}\right|$$

Q3: let $A = \begin{bmatrix} 2 & -1 & -1 \\ 3 & 1 & 0 \\ -3 & -1 & 0 \end{bmatrix}$ is A diagnolizable? if yes, then find a matrix D and an invertible matrix Q such that $Q^{-1} A Q = D$

$$C_A(\alpha) = |\alpha I_3 - A|$$

$$\begin{bmatrix} \alpha-2 & 1 & 1 \\ \boxed{-3 \quad \alpha-1 \quad 0} \\ 3 & 1 & \alpha \end{bmatrix} = \cancel{(-1)^{1+2}}^{\,-1} (-3) \begin{vmatrix} 1 & 1 \\ 1 & \alpha \end{vmatrix} + \cancel{(-1)^{2+2}}^{\,1} (\alpha-1) \begin{vmatrix} \alpha-2 & 1 \\ 3 & \alpha \end{vmatrix}$$

$$= (3)\left[(\alpha)(1) - (1)(1) \right] + (\alpha-1)\left[(\alpha-2)(\alpha) - (1)(3) \right]$$

$$= (\cancel{3\alpha - 3}) + (\alpha-1)(\alpha-2)(\alpha) - (\cancel{3\alpha-3})$$

$$= \alpha(\alpha-1)(\alpha-2) = 0$$

$\alpha = 0 \quad \alpha = 1 \quad \alpha = 2$

$$E_0 = \begin{bmatrix} \underline{\underline{-2}} & 1 & 1 \\ -3 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix} \underset{\sim}{-\tfrac{3}{2}R_1 + R_2 \to R_2} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -5/2 & -3/2 \\ 3 & 1 & 0 \end{bmatrix} \text{ etc. } \ldots$$

$E_0 = \text{span}\{(1/5, -3/5, 1)\}$

$E_1 = \text{span}\{(0, -1, 1)\}$

$E_2 = \text{span}\{(-1/3, -1, 1)\}$

$$Q = \begin{bmatrix} -1/3 & 0 & 1/5 \\ -1 & -1 & -3/5 \\ 1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & & 0 \end{bmatrix}$$

Q4: let A be a 3x3 matrix s.t. $C_A(\alpha) = \alpha^{2}(\alpha+2)$. given $E_0 = \text{span}\{(1,1,2)\}$ and $E_{-2} = \{(-1,1,0)\}$

$n_i = 2$

(a) is A diagnolizable?

no, $\dim(E_0) = 1 \neq 2$

(b) find all points in $\mathbb{R}^3$, say $Q = (a_1\ a_2\ a_3)$ s.t. $AQ^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$E_0 = \text{span}\{(1,1,2)\}$

(c) find all points in $\mathbb{R}^3$ say $Q = (a_1\ a_2\ a_3)$ s.t. $AQ^T = 5Q^T$

5 is NOT an eigen value

$\therefore = \text{span}\{(0,0,0)\}$

(d) find $A\begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}$

$$A\begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix} = -2\begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ 0 \end{bmatrix}$$

Q5: let $A = \begin{bmatrix} 2 & 2 & 3 \\ -2 & 5 & 6 \\ -2 & -2 & 7 \end{bmatrix}$ Find the $(2,3)$ entry of A^{-1}

$$= \frac{(-1)^{2+3} \begin{vmatrix} \text{del } r=3 \\ \text{col}=2 \end{vmatrix}}{|A|}$$

$|A| = ?$ $\begin{bmatrix} 2 & 2 & 3 \\ -2 & 5 & 6 \\ -2 & -2 & 7 \end{bmatrix} \underset{R_1+R_3 \to R_3}{\overset{R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 7 & 9 \\ 0 & 0 & 10 \end{bmatrix}$ $|A| = (2)(7)(10) = 140$

$$= \frac{(-1)^5 \begin{vmatrix} 2 & 3 \\ -2 & 6 \end{vmatrix}}{140} = \frac{(-1)\left[(2)(6) - (3)(-2) \right]}{140} = \frac{-18}{140}$$

same as before but instead of points, matrices

$R^{n \times n}$ = set of all matrices

Q: $R^{2\times2} = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_4 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ $\therefore \dim(R^{2\times2}) = 4$

$\begin{bmatrix} 5 & 7 \\ 1 & 2 \end{bmatrix} \in \mathbb{R}^{2\times2} = ?$

$$= 5 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 7 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Q: $R^{2\times3} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_1, \dots, a_6 \in \mathbb{R} \right\}$

$$= \left\{ a_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + a_6 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \; a_1\, a_2\, a_3\, a_4\, a_5\, a_6 \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1&0&0 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0 \\ 1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0 \\ 0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0 \\ 0&0&1 \end{bmatrix} \right\}$$

Know:

dim of $n \times m = (n)(m)$

Q: $D = \left\{ \begin{bmatrix} a+b & -1 \\ 0 & a \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$ convince me D is not a subspace of $R^{2\times2}$

SOLUTION:

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin D$ the problem is w/ -1

OR

we try to write as span we observe its not equal to a span of finite number of indep points

$$= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right\}$$

NOTE:

P_n = set of all polynomial of degree $< n$

Q: $P_3 = \{ a_2 x^2 + a_1 x + a_0 \mid a_1, a_2, a_0 \in \mathbb{R} \}$

$5 \in P_3$? yes

$2x + \sqrt{3} \in P_3$? yes

$6x^2 - \sqrt{2}x + \sqrt{11} \in P_3$? yes

$2x^3 + 1 \in P_3$? No

Polynomials

$$= \{ a_2 (x^2) + a_1 (x) + a_0 (1) \} = \text{span} \{ x^2, x, 1 \}$$

Q: find $C_0 \; C_1 \; C_2$. $C_0 + C_1 x + C_2 x^2 = 0$

$= 0 + 0x + 0x^2$

$C_0 = 0 \quad C_1 = 0 \quad C_2 = 0 \qquad \dim = 3$

Know:

$\dim(P_n) = n$

Q: convince me that $D = \{ a_0 + a_2 x + a_2 x^2 \mid a_0 \; a_2 \in \mathbb{R} \}$ is a subspace

$$\{ a_0 (1) + a_2 (x + x^2) \} = \text{span} \{ (1), (x + x^2) \}$$

Know:

$\mathbb{R}^{n \times m} \approx \mathbb{R}^{nm}$

isomorphic

same as subspaces

$\mathbb{R}^{2\times 2} \approx \mathbb{R}^4$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longleftrightarrow (1,2,3,4)$

Q: translate the two points to matrices

$T = (1,2,3,4)$ $\quad T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = (-1,0,0,1)$

Q: $D = \text{span}\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \right\}$

(1) find dim (D)

so, its going to be in the co-space ($\mathbb{R}^4$) of $\mathbb{R}^{2\times 2}$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \longleftrightarrow (1,2,0,1)$ $\quad \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \longleftrightarrow (-1,-1,1,1)$ $\quad \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \leftrightarrow (1,3,1,3)$

now, write the point in the matrix and kill before to check if the points are dep or indep

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix} \underset{-R_1+R_3 \to R_3}{\overset{R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \overset{-R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \text{indep}$$

$(1,2,0,1)\ (0,1,1,2) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ are indep

dim (D) = 2

(2) write D as span of basis

basis of D = $\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right\}$

D = {span of basis} = $\text{span}\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right\}$

Q: Find a basis for $\mathbb{R}^{2\times 2}$, say B, s.t.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \in B \qquad \dim = 2\times 2 = 4$$

Solution: consider the co-space $\mathbb{R}^4$

$\mathbb{R}^{2\times 2}$ $\quad$ $\mathbb{R}^4$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \longleftrightarrow (1,1,1,1)$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \longleftrightarrow (-1,-1,1,1)$$

$$\begin{bmatrix} \underline{\underline{1}} & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ & & & \\ & & & \end{bmatrix} \overset{R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

need to add 2 points

by staring LOL

$$B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

fact:

$P_n \cong \mathbb{R}^n$ isomorphic as spaces

Q: $P_4 \longleftrightarrow \mathbb{R}^4$

$a_3x^3 + a_2x^2 + a_1x + a_0 \longleftrightarrow (a_3, a_2, a_1, a_0)$

$2x^3 - 10x + 15 \longleftrightarrow (2, 0, -10, 15)$

$13x^2 + 2x - 10x^3 + 15 \longleftrightarrow (-10, 13, 2, 15)$

Q: $D = \{ (a_2+a_1)x^3 + a_2x^2 - a_1x + a_1 \mid a_1, a_2 \in \mathbb{R} \}$

D "lives" in P_4

(a) Convince me that D is a subspace of P_4

(b) find a basis for P_4

(a) $D = \{ a_2(x^3+x^2) + a_1(x^3 - x + 1) \}$

$= \text{span}\{ (x^3+x^2), (x^3-x+1) \}$

to check for independent, we use the co-space, $\mathbb{R}^4$

P_4 $\quad$ $\mathbb{R}^4$

$x^3+x^2 \longleftrightarrow (1,1,0,0)$

$x^3-x+1 \longleftrightarrow (1,0,-1,1)$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} \overset{-R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

by staring, you can see that they're independent

(b) Basis = $\{ x^3 + x^2 \, , \; x^3 - x + 1 \}$

domain

co-domain

Q: $T: \mathbb{R}^{2\times 2} \longrightarrow P_3$

$$T \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = (a_1 + a_4)\, x^2 + a_1 x + a_4$$

⚠ this Q coming in the MD

① Convince me that T is a Linear Transformation

② find all matrices in $\mathbb{R}^{2\times 2}$ s.t. $T \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = x^2 - x + 3$

③ find the $Z(T)$, i.e. set of all matrices in $\mathbb{R}^{2\times 2}$ s.t. $T \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = 0$

SOLUTION:

① find the co-linear transformation of T

$L: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$$L(a_1 \; a_2 \; a_3 \; a_4) = \underbrace{(a_1 + a_4 \, , \; a_1 \, , \; a_4)}_{\text{each coord is a linear comb of } a_1 \; a_2 \; a_3 \; a_4}$$

note: fake matrix presentation is co-matrix co-domain?

$\therefore$ L is a LT $\approx$ T is a L.T.

② He didn't finish solving

domain

co-domain

Q: $T: P_3 \to \mathbb{R}^3$

$$T(a_2 x^2 + a_1 x + a_0) = (a_2 + a_1 + a_0 \, , \, a_1 \, , \, a_0)$$

is T a LT? yes

find the co-matrix presentation of T. OLD NOTES SAYS FAKE MATRIX PRES.

SOLUTION

$L: \mathbb{R}^3 \to \mathbb{R}^3$

$L(a_2, \, a_1, \, a_0) = (a_2 + a_1 + a_0, \; a_1, \; a_0)$

co-matrix presentation of T is the matrix pres of L

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is T invertible ?

T is invertible if L is invertible and M is invertible

Find M^{-1} :

$$\left[\begin{array}{ccc|ccc} \underline{\underline{1}} & 1 & 1 & 1 & 0 & 0 \\ 0 & \underline{\underline{1}} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{-R_2+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{-R_3+R_1 \to R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\underbrace{\quad\quad}_{I_3}$ $\underbrace{\quad\quad}_{M^{-1}}$

$\therefore$ T is invertible

Find T^{-1} :

$L^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$L^{-1}(a_1\ a_2\ a_3) = M^{-1}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = (a_1-a_2-a_3,\ a_2,\ a_3)$$

$T^{-1} : \mathbb{R}^3 \to P_3$

$$T^{-1}(a_1, a_2, a_3) = (a_1 - a_2 - a_3)\ x^2 + a_2 x + a_3$$

$T^{-1}(1,1,0)$? you just substitute

(a_1, a_2, a_3)

$$= (1-1-0)\ x^2 + (1)\ x + 0 \quad = x$$

What are the $Z(T)$?

$$\dim(\text{Range}) + \dim(Z) = \dim(\text{domain})$$

$$3 + 0 = 3$$

$$Z(T) = \{(0,0,0)\}$$

Result (Know):

- A linear transformation
  T is 1-1 <u>IF</u>
  $Z(T) = \{0\text{-element}\}$

- As soon as you know D is a subspace
  $\dim(D) < \infty$. Then, the following must hold:

  (1) $\forall a, b \in D$, $a+b \in D$ ← closed under addition

  (2) $\forall c \in \mathbb{R}$ and $a \in D$, $ca \in D$ ← closed under scalar multiplication

<u>Q</u>: convince me that

$$D = \left\{ \begin{bmatrix} a+b & a \\ a & a+1 \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}$$

Show that you can write it as span

NOT a subspace

$a(1,1,1,1) + b(1,0,0,0) + (0,0,0,1)$

<u>OR</u> by staring

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin D$$

<u>Q</u>: (1) $D = \{ A \in \mathbb{R}^{3\times3} \mid |A| = 0 \}$

convince me that D is not a subspace of $\mathbb{R}^{3\times3}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in D$$

$\forall a, b \in D$ $a+b \in D$ is false

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|a| = 0 \quad \therefore a \in D$ $\qquad$ $|b| = 0 \quad \therefore b \in D$

$$a+b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad |a+b| = 1 \quad \therefore a+b \notin D$$

Q: $D = \{ f(x) \in P_3 \mid f(0) = 0 \text{ OR } f(1) = 0 \}$

Show that D is not a subspace. D "lives" inside P_3

$f_1(x) = x \in D$ (why?) $f(0) = 0$

$f_2(x) = 1 - x \in D$ (why?) $f_2(1) = 0$

$f_3 = f_1 + f_2 = 1 \notin D$ (why?) $f_3(0) \neq 0$ $f_3(1) \neq 0$

closure under addition FAILS

Show that $D = \{ A \in \mathbb{R}^{2\times2} \mid A^T = -A \}$ is a subspace

Show (1) closure under addition and (2) closure under scalar multiplication

NOTE: You can ANSW this Q in two ways:
(1) axioms
(2) find a set of finite points in span

$\boxed{1}$ ① let $a, b \in D$

show that $a + b \in D$

$\underbrace{a^T = -a \qquad b^T = -b}$

why? bcuz $a, b \in D$

$(a+b)^T = a^T + b^T = -a + -b = -(a+b)$

② let $a \in D$ & $c \in \mathbb{R}$ show $ca \in D$

$a^T = -a$ (why?) bcuz $a \in D$

$(ca)^T = ca^T = -ca \Rightarrow ca \in D$

OR

$\boxed{2}$ $D = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \underbrace{\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{bmatrix}} \right\}$

by staring

$a_1 = -a_1 \quad \therefore a_1 = 0$

$a_3 = -a_2$] → same

$a_2 = -a_3$

$a_4 = -a_4 \quad \therefore a_4 = 0$

$D = \left\{ \begin{bmatrix} 0 & a_2 \\ -a_2 & 0 \end{bmatrix} \middle| a_2 \in \mathbb{R} \right\}$

$D = \left\{ a_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \middle| a_2 \in \mathbb{R} \right\}$

$D = \text{span} \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

prof likes the lang. of span more

Find $\underline{\dim(D)} = 1$

in prev exams

$\text{IN}(D) = 1$

Q: $D = \{ f(x) \in P_3 \mid f(0) = 0 \text{ AND } f(1) = 0 \}$

Show D is a subspace

Find dim (D)

$$D = \{ \underbrace{a_2 x^2 + a_1 x + a_0}_{f(x)} \mid \underbrace{f(0) = a_0 = 0 \quad \text{AND} \quad f(1) = \underbrace{a_2 + a_1}_{a_1 = -a_2} = 0}_{\text{conditions}} \}$$

$$D = \{ a_2 x^2 - a_2 x + 0 \mid a_2 \in \mathbb{R} \}$$
$$= \{ a_2 (x^2 - x) \mid a_2 \in \mathbb{R} \}$$
$$= \text{span} \{ (x^2 - x) \}$$

$\dim(D) = 1$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \overset{2R_1}{\sim} \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} \overset{R_1 \; R_2}{\longleftrightarrow} \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix} \overset{2R_1 + R_2 \to R_2}{\sim} \underset{B}{\begin{bmatrix} 3 & 0 \\ 8 & 4 \end{bmatrix}}$$

all of these are eq into each other

Q: Find 3 elementary matrices E_1, E_2, E_3 s.t. $E_1 E_2 E_3 A = B$

$E_3 = 2R_1$

$E_2 = R_1 \leftrightarrow R_2$

$E_1 = 2R_1 + R_2 \to R_2$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{2R_1}{\sim} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\overset{2R_1 + R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$I_2 \; A_{2\times2}$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A_{2\times2}$$

$$\underset{3\times4}{A} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

$$\overset{-2R_3}{\sim} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & -2 & 0 & -6 \end{bmatrix} = B$$

Find elementary matrices E_1, E_2 s.t. $E_1 E_2 \underset{3\times4}{A} = B$

$$E_2 = I_3 \overset{R_1 \leftrightarrow R_2}{\sim} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_1 = I_3 \overset{-2R_3}{\sim} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

DOT PRODUCT:

Dot product over $\mathbb{R}^n$

$(a_1, \dots, a_n)\,(b_1, \dots, b_n)$

$= a_1b_1 + a_2b_2 + \dots + a_nb_n \in \mathbb{R}$

Def:

$\mathbb{R}^2$ standard basis = $\{(1,0)\,(0,1)\}$

$\mathbb{R}^4$ standard basis = $\{(1,0,0,0)\,(0,1,0,0)\,(0,0,1,0)\,(0,0,0,1)\}$ → orthogonal basis

meaning dot product = 0

let $Q_1, Q_2, \dots, Q_m \in \mathbb{R}^n$

we say $Q_1, \dots, Q_m$ are orthogonal

<u>IF</u> $Q_i \cdot Q_K = 0 \quad i \neq K$

dot product

<u>Q:</u> convince me $\{(1,2)\,(0,4)\}$ is a basis for $\mathbb{R}^2$

$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ by staring, you can see theyre indep

other basis:

$\{(1,0)\,(0,1)\}$

$\{(4,10)\,(0,13)\}$

$\mathbb{R}^2 = \text{span}\{\underset{Q_1}{(1,2)}\,\underset{Q_2}{(0,4)}\}$

$= \text{span}\{(4,10)\,(0,13)\}$

$c_1Q_1 + c_2Q_2 = (0,0)$

$c_1 = c_2 = 0$

WORKSHEET:

<u>Q1:</u> let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^{2\times 2}$ s.t.

$$T\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = \begin{bmatrix} a_1 - 2a_4 & 2a_3 + a_2 \\ -a_1 + 4a_3 & a_2 + 4a_1 \end{bmatrix}$$

(i) show that T is a linear transformation

by staring, you can see that $\begin{bmatrix} a_1 - 2a_4 & 2a_3 + a_2 \\ -a_1 + 4a_3 & a_2 + 4a_1 \end{bmatrix}$ is a linear comb. of $a_1\ a_2\ a_3\ a_4$

$\therefore$ T is a linear transformation

(ii) find Range (T)

$$\text{Range}(T) = \left\{ \begin{bmatrix} a_1 - 2a_4 & 2a_3 + a_2 \\ -a_1 + 4a_3 & a_2 + 4a_1 \end{bmatrix} \middle| \; a_1\, a_2\, a_3\, a_4 \in \mathbb{R} \right\}$$

$$= \left\{ a_1 \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} + a_4 \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

(iii) find dim (Range (T))

$$\begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 4 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{2R_1 + R_4 \to R_4} \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & -2 & 8 \end{bmatrix} \underset{\sim}{-2R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & -2 & 8 \end{bmatrix} \underset{\sim}{\text{etc.}} \quad \dim(R) = 4$$

(iv) find a basis for Range (T)

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

NOTE:

Range (T) = $\mathbb{R}^{2\times2}$ Hence T is ONTO thus any 4 indep matrices will form a basis for Range (T)

(v) Find Z(T)

dim (Range) + dim (Z) = dim (D)

$4 + 0 = 4$

dim (Z) = 0

$$Z(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

Q2: Let $T: P_2 \to \mathbb{R}^2$ s.t.

$$T(a_1 x + a_0) = (2a_1 + 11a_0 \, , \; a_1 + 5a_0)$$

Convince me that T is invertible. Find $T^{-1}(4,7)$

note:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$L: \mathbb{R}^2 \to \mathbb{R}^2$

$$L(a_1, a_0) = (2a_1 + 11a_0 \, , \; a_1 + 5a_0)$$

$$= (a_0 (11, 5) + a_1 (2, 1))$$

$$m = \begin{array}{c} \begin{array}{cc} a_1 & a_0 \end{array} \\ \begin{bmatrix} 2 & 11 \\ 1 & 5 \end{bmatrix} \end{array} \longrightarrow m^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -11 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 11 \\ 1 & -2 \end{bmatrix}$$

$$|A| = (2)(5) - (11)(1) = 10 - 11 = -1$$

$L^{-1}(a_1\ a_2) = \begin{bmatrix} -5 & 11 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = (-5a_1+11a_2\ ,\ a_1-2a_2)$

$T^{-1}: \mathbb{R}^2 \to P_2$

$T^{-1}(a_1\ a_2) = (-5a_1+11a_2)x_1 + (a_1-2a_2)$

$T^{-1}(4,7) = (-5(4)+11(7))\ x_1 + (4-2(7))$

$= 57x_1 - 10$

Q3: let

$F = \{ (a+b)x^3 + ax^2 + ax + (a-b) \mid a,b \in \mathbb{R} \}$

show that F is a subspace

$= \{ a(x^3+x^2+x+1) + b(x^3-1) \}$

$= \text{span}\{ (x^3+x^2+x+1)(x^3-1) \}$

worksheet:

Q1: let $T: \mathbb{R}^3 \to P_3$ s.t. $T(a_1\ a_2\ a_3) = (a_1+a_3)x^2 + a_3x + a_1$

(i) find the co-matrix presentation of T

$L: \mathbb{R}^3 \to \mathbb{R}^3$

$L(a_1\ a_2\ a_3) = \{ (a_1+a_3)(a_3)(a_1) \} = \{ a_1(1,0,1) + a_2(0,0,0) + a_3(1,1,0) \}$

$$\begin{array}{ccc} a_1 & a_2 & a_3 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$= \text{span}\{ (1,0,1)(0,0,0)(1,1,0) \}$

(ii) find a basis for the range of T

$= \{ (a_1+a_3)x^2 + a_3x + a_1 \}$

$= \{ a_1(x^2+1) + a_2(0) + a_3(x^2+x) \}$

Range $= \text{span}\{ (x^2+1)(x^2+x) \}$

$B: \{ (x^2+1)(x^2+x) \}$

(iii) find all points in $\mathbb{R}^3$ s.t. $T(a_1\ a_2\ a_3) = x^2+1$

from (1) we can rephrase

$T(a_1\ a_2\ a_3) = x^2+1 = (1,0,1)$

$\therefore$ we form this aug. matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{-R_1+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \begin{array}{c} -R_2+R_1\to R_1 \\ \sim \\ R_2+R_3\to R_3 \end{array} \begin{array}{cccc} a_1 & a_2 & a_3 & c \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$a_1 = 1$
$a_3 = 0$

$\therefore$ a_1 & a_3 leading var $= \{(1, a_2, 0) \mid a_2 \in \mathbb{R}\}$
a_2 free var.

(iv) is T invertible? if yes, find T^{-1} and find $T^{-1}(2x^2+x)$

$L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$L(a_1, a_2, a_3) = \{(a_1+a_3)\ (a_3)\ (a_1)\}$

$= \{a_1(1,0,1) + a_2(0,0,0) + a_3(1,1,0)\}$

$= \text{span}\{(1,0,1)\ (0,0,0)\ (1,1,0)\}$

note:

if M is invertible
and L is invertible
then T is invertible

$$\therefore M = \begin{array}{ccc} a_1 & a_2 & a_3 \\ \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] \end{array} \rightarrow [M \mid I_3]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{-R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} -R_2+R_1\to R_1 \\ \sim \\ R_2+R_3\to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

$\downarrow$
$\neq I_3$

$\therefore$ M is not invertible so L is not invertible and T is not invertible

(v) let $H = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$ show that H is a subspace and find $\dim(H)$

$$= \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \;\middle|\; \underbrace{\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}}_{\text{by staring}} \right\}$$

$a_1 = a_1$
$a_3 = a_2$
$a_2 = a_3$
$a_4 = a_4$

$$\rightarrow \left\{ \underbrace{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \;\middle|\; a_2 = a_3,\ a_4, a_1 \in \mathbb{R}}_{\begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix} \mid a_2, a_4, a_1 \in \mathbb{R}} \right\}$$

$$= \left\{ \begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix} \middle| \; a_2, a_1, a_4 \in \mathbb{R} \right\}$$

$$= \left\{ a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} \underline{\underline{1}} & 0 & 0 & 0 \\ 0 & \underline{\underline{1}} & 1 & 0 \\ 0 & 0 & 0 & \underline{\underline{1}} \end{bmatrix} \xrightarrow{\text{now we kill}} \text{below to make sure theyre INDEP}$$

$\therefore$ theyre all indep

dim (D) = 3

(vi) let

$F = \{ (2a+6b)x^3 + (-a-3b)x^2 + (a+3b) \}$

show that F is a subspace and find dim (F)

L: $\{ (2a+6b)(-a-3b)(a+3b) \}$

$= \{ a(2,-1,1) + b(6,-3,3) \}$

$= \text{Span}\{ (2,-1,1)(6,-3,3) \}$

$$m = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} \underline{\underline{2}} & 6 \\ -1 & -3 \\ 1 & 3 \end{bmatrix} \end{array} \underset{-\frac{1}{2}R_1 + R_3 \to R_3}{\overset{\frac{1}{2}R_1 + R_2 \to R_2}{\sim}} \begin{bmatrix} 2 & 6 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$F = \{ a(2x^3 - x^2 + 1) + b(6x^3 - 3x^2 + 3) \mid a, b \in \mathbb{R} \}$

$= \text{span}\{ (2x^3 - x^2 + 1)(6x^3 - 3x^2 + 3) \}$

$\therefore$ F is a subspace of finite points

dim (F) = 1

Q_1 Q_2

Q: $D = \text{span}\{(1,2,1)\ (-1,1,1)\}$

D lives in $\mathbb{R}^3$

$\dim(D) = 2$

find an orthogonal basis of D

· should be 2 indep points

· should be 2 points where their dot product is 0

gram - schmitt - algorithm

dim(D) = 2 so, 2 points

$O = \{w_1, w_2\}$

$w_1 = Q_1 = (1,2,1)$

$$w_2 = Q_2 - \frac{Q_2 \cdot Q_1}{|Q_1|^2} Q_1$$

$Q_1, Q_2 \in \mathbb{R}^n$

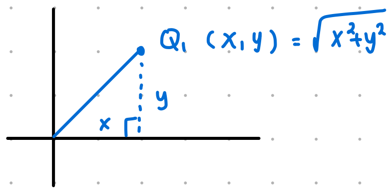
$(a_1, \dots, a_n)$ $(b_1, \dots b_n)$

$Q_1 \cdot Q_2 = a_1 b_1 + \cdots + a_n b_n$

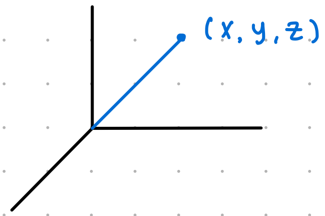
$|Q_1| = \sqrt{a_1^2 + \cdots + a_n^2}$

$|Q_1|^2 = a_1^2 + \cdots + a_n^2$

2D:



3D:



$$w_2 = (-1,1,1) - \frac{2}{1^2+2^2+1^2}(1,2,1)$$

$$= (-1,1,1) - \frac{2}{6}(1,2,1) \quad \frac{1}{3}$$

$= (-1,1,1) - (1/3, 2/3, 1/3)$

$= (-4/3, 1/3, 2/3)$

to check:

$w_1 \cdot w_2 = (1,2,1) \cdot (-4/3, 1/3, 2/3)$

$$= -\frac{4}{3} + \frac{2}{3} + \frac{2}{3} = 0$$

$O = \{w_1, w_2\}$

$= \{(1,2,1)\ (-4/3, 1/3, 2/3)\}$

$D = \text{span}\{\underbrace{Q_1, \dots, Q_K}_{\text{indep. points}}\}$ $\dim(D) = K$

find an orthogonal basis of D.

$O = \{W_1, \dots, W_K\}$

$W_1 = Q_1$

$W_2 = Q_2 - \frac{Q_2 \cdot W_1}{|W_1|^2} W_1$

$W_3 = Q_3 - \frac{Q_3 \cdot W_1}{|W_1|^2} W_1 - \frac{Q_3 \cdot W_2}{|W_2|^2} W_2$

V.I.P

$$\boxed{W_m = Q_m - \frac{Q_m \cdot W_1}{|W_1|^2} W_1 - \dots - \frac{Q_m \cdot W_{m-1}}{|W_{m-1}|^2} W_{m-1}}$$

Result:

- if $Q_1, Q_2, \dots, Q_K$ are non-zero points in $\mathbb{R}^n$ and orthogonal, then $Q_1, \dots, Q_K$ are independent
  ↳ INDEP is NOT always orthogonal

Q: $Q_1 = (2,4)$ are they orthogonal?
$Q_2 = (-2,4)$

NO! dot product $= (2)(-2) + (4)(4) = 12$

$\therefore$ Q_1 & Q_2 are not orthogonal

BUT Q_1 & Q_2 are INDEP ⟶ $Q = \begin{bmatrix} 2 & 4 \\ -2 & 4 \end{bmatrix}$

$|Q| = (2)(4) - (-2)(4) = 16$

$\therefore$ $|Q| \neq 0$

How do we find the orthoginal basis?
↳ we use the gram-schmitt alg.

inner product on polynomials:

$$<f_1, f_2> = \int_a^b f_1 f_2 \, dx$$

Q: $D = \text{span}\{1, x^2+1\} \subseteq P_3 \longrightarrow \dim(D) = 2$

find orthogonal basis for D where $<f_1, f_2> = \int_0^1 f_1 f_2 \, dx$

To find basis: $O = \{w_1, w_2\}$

you do either $\longrightarrow <w_1, w_2> = 0$

$\searrow \int_0^1 w_1 w_2 = 0$

lets check if f_1 f_2 are orthogonal:

$$\int_0^1 (1)(x^2+1)\,dx = \frac{1}{3}x^3 + x \Big|_0^1 = \frac{4}{3} \neq 0 \quad \therefore \text{ not orthogonal}$$

so, we have to find basis that are orthogonal:

if f is a polynomial $|f| = \sqrt{\int_a^b f^2 dx}$

so, the norm is $|f|^2 = \int_a^b f^2 \, dx$

$w_1 = Q_1 = f_1 = 1$

$$w_2 = f_2 - \frac{\int_0^1 f_1 f_2 \, dx}{|f_1|^2} \cdot f_1$$

$$= (x^2+1) - \frac{\int_0^1 (x^2+1)(1)\,dx}{|f_1|^2} f_1$$

$$= (x^2+1) - \frac{\int_0^1 (x^2+1)(1)\,dx}{\int_0^1 1\,dx} \cdot 1$$

$$= x^2 + 1 - 4/3 = x^2 - 1/3$$

now:

$O = \{1, x^2 - 1/3\}$

$O = \text{span}\{1, x^2 - 1/3\}$ $\leftarrow$ span of orthogonal basis

to check: integrate orthogonal basis
and you'll get 0

$f_1 \quad f_2 \quad f_3$

Q: $D = \text{span}\{x, x^2, x^4\}$ "lives" in P_5

inner product on D is defined

$\langle f_1\, f_2 \rangle = \int_0^1 f_1 f_2\, dx$

find $O = \{w_1, w_2, w_3\}$

$w_1 = f_1 = x$

$$w_2 = f_2 - \frac{\int_0^1 w_1 f_2\, dx}{|w_1|^2} w_1$$

$$w_3 = f_3 - \frac{\int_0^1 w_2 f_3}{|w_2|^2} \cdot w_2 - \frac{\int_0^1 w_1 f_3}{|w_1|^2} \cdot w_1$$

for w_2:

$$\int_0^1 w_1 f_2\, dx = \int_0^1 x\, x^2\, dx = 1/4$$

$$|w_1|^2 = \int_0^1 w_1^2\, dx = \int_0^1 x^2\, dx = 1/3$$

$$w_2 = x^2 - \frac{1/4}{1/3} x = x^2 - 3/4\, x$$

for w_3:

expand

- $\int_0^1 w_2 f_3\, dx = \int_0^1 (x^2 - 3/4\, x)(x^4)\, dx$

$$= \int_0^1 x^6 - 3/4\, x^5\, dx = \int_0^1 x^6\, dx - 3/4 \int_0^1 x^5\, dx$$

$$= 1/7 - 1/8 = 1/56$$

expand

- $|w_2|^2 = \int_0^1 w_2^2\, dx = \int_0^1 (x^2 - 3/4\, x)^2\, dx$

$$= \int_0^1 x^4 - \frac{3}{2} x^3 + \frac{9}{16} x^2\, dx$$

$$= \int_0^1 x^4\, dx - 3/2 \int_0^1 x^3\, dx + 9/16 \int_0^1 x^2\, dx$$

$$= 1/5 - 3/8 + 3/16 = 1/80$$

- $\int_0^1 w_1 f_3\, dx = \int_0^1 (x)(x^4)\, dx = \int_0^1 x^5\, dx$

$$= \frac{1}{6} x^6 \Big|_0^1 = 1/6 - 0 = 1/6$$

- $|w_1|^2 = 1/3$

$$w_3 = f_3 - \frac{\int_0^1 w_2 f_3}{|w_2|^2} \cdot w_2 - \frac{\int_0^1 w_1 f_3}{|w_1|^2} \cdot w_1$$

$$= x^4 - \frac{1/56}{1/80}\left(x^2 - 3/4\, x\right) - \frac{1/6}{1/3}\, x$$

" All subspaces in MTH 221 are called vector spaces "

set vector

$(V, +, \cdot)$ is called a vector space if :

addition

scalar multiplication

[1] $\forall\ x, y \in V$, $x + y \in V$ closed under addition

[2] $\forall c \in \mathbb{R}$ and $\forall x \in V$ so $cx \in V$ closed under multiplication

Use rule [1] or [2] if you want to prove it is not a vector space
prove one of them wrong !

[3] $\exists$ zero element in V, call it 0

[4] $\forall x \in V$, $\exists\ -x \in V$

[5] $\forall\ c_1, c_2 \in \mathbb{R}$, $\forall x \in V$ $(c_1 + c_2)x = c_1 x + c_2 x$

[6] $\forall\ c_1, c_2 \in \mathbb{R}$, $\forall x \in V$ $(c_1 c_2)x = c_1 (c_2 x)$

[7] $\forall\ c \in \mathbb{R}$ $x, y \in V$ $c(x+y) = cx + cy$

$$f(x) = \frac{1}{x} \in D$$

$D = C[1,2]$ set of all continuous function on $[1,2]$
this is a subspace BUT cant be written as span

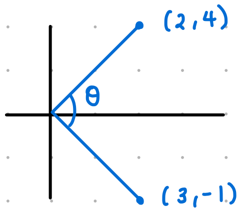
$\dim(D) = \infty$

solution: let $f_1, f_2 \in D$ (f_1 f_2 are cont on $[1,2]$)

from calc 1 $f_1 + f_2$ is cont on $[1,2]$

proves axiom 1

Application #1

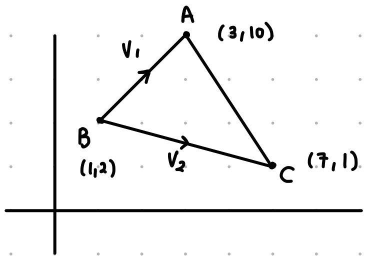


$$\cos\theta = \frac{(2,4)\cdot(3,-1)}{|(2,4)||(3,-1)|}$$

$$= \frac{6-4}{\sqrt{20}\sqrt{10}} = \frac{2}{\sqrt{200}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{200}}\right) = 81.8^\circ$$

Application #2:



Find the area of ABC

→ it is crucial for V_1 & V_2 to have the same initial point

$$V_1 = (\Delta X, \Delta Y) = (3-1, 10-2) = (2, 8)$$
$$V_2 = (\Delta X, \Delta Y) = (7-1, 1-2) = (6, -1)$$

$$\therefore \text{Area} = \left| \frac{\begin{vmatrix} 2 & 8 \\ 6 & -1 \end{vmatrix}}{2} \right| = \left| \frac{-2-48}{2} \right| = 25$$

KNOW:

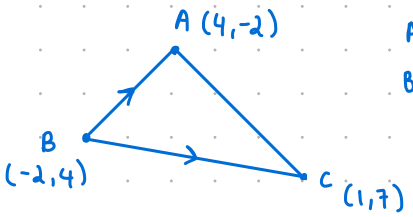
if $s_1, \dots, s_k$ are nonzero orthogonal elements of an inner product space, then they are independent

the converse need to be true

<u>ex:</u> using the dot product on $\mathbb{R}^2$, the points $(1,1)$ $(2,1)$ are independent, but they are not orthogonal

Worksheet:

<u>Q1:</u> given $A = (4,-2)$ $B(-2,4)$ $C = (1,7)$ find the area of the triangle ABC



$$AB \to V_1 = (\Delta X, \Delta Y) = (4-(-2), -2-4) = (6,-6)$$
$$BC \to V_2 = (\Delta X, \Delta Y) = (-2-1, 4-7) = (-3,-3)$$

$$\text{Area} = \left| \frac{\begin{vmatrix} 6 & -6 \\ -3 & -3 \end{vmatrix}}{2} \right| = \frac{36}{2} = 18$$

Q2: Use the inner product $\langle\ ,\ \rangle$ on P_5 such that $\langle f_1, f_2 \rangle = \int_0^1 f_1 f_2 \, dx$
find the angle between $f_1 = 1$ and $f_2 = 3x^2$

$$\theta = \cos^{-1}\left(\frac{|\langle f_1, f_2 \rangle|}{|f_1|\ |f_2|} \right)$$

note: $|f_1| = \sqrt{f_1^2}$

$$= \cos^{-1}\left(\frac{\int_0^1 (1)(3x^2)\,dx}{\sqrt{\int_0^1 (1)^2 dx}\ \sqrt{\int_0^1 (3x^2)^2 dx}} \right)$$

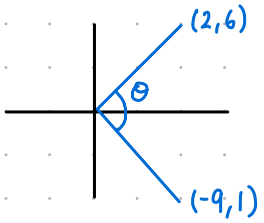
$$\int_0^1 3x^2 dx = \frac{3x^3}{3}\Big|_0^1 = x^3\Big|_0^1 = 1 - 0$$

$$\int_0^1 1\,dx = x\Big|_0^1 = 1 - 0$$

$$\int_0^1 (3x^2)^2 dx = \int_0^1 9x^4\,dx = \frac{9x^5}{5}\Big|_0^1 = 9/5 - 0$$

$$= \cos^{-1}\left(\frac{1}{3/\sqrt{5}} \right) = \cos^{-1}\left(\sqrt{5}/3\right) = 41.81^\circ$$

Q3: Use the dot product on $\mathbb{R}^2$ and find the angle b/w
$Q_1 = (2,6)$ and $Q_2 = (-9,1)$



$$\theta = \cos^{-1}\left(\frac{|\langle f_1, f_2 \rangle|}{|f_1|\ |f_2|} \right) = \cos^{-1}\left(\frac{(2,6)\cdot(-9,1)}{|(2,4)|\ |(-9,1)|} \right)$$

$$= \cos^{-1}\left(\frac{|(2,6)\cdot(-9,1)|}{\sqrt{2^2+6^2}\ \sqrt{9^2+1}} \right) = \cos^{-1}\left(\frac{|(2)(-9)+(6)(1)|}{\sqrt{40}\ \sqrt{82}} \right)$$

$$= \cos^{-1}\left(\frac{12}{4\sqrt{205}} \right) = 77.90^\circ$$

Q4: let $D = \text{span}\{2x, 3x^2, 5x^4\}$ then D is a subspace of P_5. Find an orthogonal basis of D, use the inner product $\langle f_1, f_2 \rangle = \int_0^1 f_1 f_2\,dx$

$f_1 = W_1 = 2x$ $\quad f_2 = 3x^2$ $\quad f_3 = 5x^4$

$$W_2 = f_2 - \frac{\langle W_1, f_2 \rangle}{|W_1|^2} W_1$$

$$= 3x^2 - \frac{\int_0^1 (2x)(3x^2)}{\int_0^1 (2x)^2}\ 2x$$

$$\cdot \int_0^1 (2x)(3x^2) = \int_0^1 6x^3$$

$$= \frac{6}{4} x^4 \Big|_0^1 = \frac{6}{4} - 0 = 6/4$$

$$\cdot \int_0^1 (2x)^2 = \int_0^1 4x^2 = 4/3\, x^3 \Big|_0^1 = 4/3$$

$$= 3x^2 - \frac{6/4}{4/3} 2x = 3x^2 - 9/4\, x$$

$$\boxed{\therefore \quad W_2 = 3x^2 - 9/4\, x}$$

$$W_3 = f_3 - \frac{\langle W_2, f_3 \rangle}{|W_2|^2} W_2 - \frac{\langle W_1, f_3 \rangle}{|W_1|^2} W_1$$

$$\langle W_2, f_3 \rangle = \int_0^1 (3x^2 - 9/4\, x)(5x^4) = 15/56$$

$$|W_2|^2 = \int_0^1 (3x^2 - 9/4\, x)^2 = 9/80$$

$$|W_1|^2 = 4/3$$

$$\langle W_1, f_3 \rangle = \int_0^1 (5x^4)(2x) = 5/3$$

$$W_3 = 5x^4 - \frac{15/56}{9/80} (3x^2 - 9/4\, x) - \frac{5/3}{4/3} 2x$$

$$= 5x^4 - 50/21 (3x^2 - 9/4\, x) - 5/2\, x$$

$$= 5x^4 - \frac{50}{7} x^2 - 75/14\, x - 5/2\, x$$

$$\boxed{W_3 = 5x^4 - 50/7\, x^2 - 20/7\, x}$$

Q5: let $D = \text{span}\{(1,0,1,1)\ (1,0,0,0)\ (1,0,0,1)\}$. Find an orthogonal basis of D, use the dot product $\langle f_1, f_2 \rangle = f_1 \cdot f_2$

$f_1 = w_1 = (1,0,1,1)$

note.

$Q_1 \cdot Q_2 = a_1 b_1 + \cdots + a_n b_n$

$|Q_1|^2 = a_1^2 + \cdots + a_n^2$

$f_2 = (1,0,0,0)$

$f_3 = (1,0,0,1)$

$0 : \{w_1, w_2, w_3\}$

$$w_2 = f_2 - \frac{\langle w_1, f_1 \rangle}{|w_1|^2} w_1$$

$\langle w_1, f_1 \rangle = (1,0,1,1)\cdot(1,0,0,0) = 1$

$|w_1|^2 = 1^2 + 0^2 + 1^2 + 1^2 = 3$

$w_2 = (1,0,0,0) - 1/3\,(1,0,1,1)$

$= (1,0,0,0) - (1/3, 0, 1/3, 1/3)$

$= (2/3, 0, -1/3, -1/3)$

$$w_3 = f_3 - \frac{\langle w_2, f_3 \rangle}{|w_2|^2} w_2 - \frac{\langle w_1, f_3 \rangle}{|w_1|^2} w_1$$

$\langle w_2, f_3 \rangle = (2/3, 0, -1/3, -1/3)\cdot(1,0,0,1) = (2/3)(1) + (-1/3)(1) = 1/3$

$|w_2|^2 = (2/3)^2 + 0^2 + (-1/3)^2 + (-1/3)^2 = 2/3$

$\langle w_1, f_3 \rangle = (1,0,1,1)\cdot(1,0,0,1) = 2$

$|w_1|^2 = 3$

$w_3 = (1,0,0,1) - \frac{1/3}{2/3}(2/3, 0, -1/3, -1/3) - 2/3\,(1,0,1,1)$

$= (1,0,0,1) - 1/2\,(2/3, 0, -1/3, -1/3) - (2/3, 0, 2/3, 2/3)$

$= (1,0,0,1) - (1/3, 0, -1/6, -1/6) - (2/3, 0, 2/3, 2/3)$

$$= \begin{bmatrix} 1 - 1/3 - 2/3 \\ 0 - 0 - 0 \\ 0 - (-1/6) - 2/3 \\ 1 - (-1/6) - 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$0 = \{ (1,0,1,1), (2/3, 0, -1/3, -1/3), (0, 0, -1/2, 1/2) \}$

Check.

$= (1)(2/3)(0) + (0)(0)(0) + (1)(-1/3)(-1/2) + (1)(-1/3)(1/2)$

$= 0 + 0 + 1/6 - 1/6 = 0$

Q6: find a nonzero point $Q = (a\ b\ c)$ such that Q is orthogonal to $(1,2,2)$ and $|Q| = 2022$

Formula: $\frac{|c|}{|m|} m = c$

$m\,(1,2,2) = 0$ orthogonal if dot product is 0

$(a,b,c)\cdot(1,2,2) = 0$

$1a + 2b + 2c = 0$

now choose any a, b, c that satisfies the eq above

$a = -4 \quad b = 1 \quad c = 1$

$(1)(-4) + 2(1) + 2(1) = 0$

$0 = 0$

so, one orthogonal base is $m = (-4, 1, 1)$

but $|m| = \sqrt{(-4)^2 + 1^2 + 1^2} = \sqrt{18}$

but we want orthogonal basis Q that has $|Q| = 2022$

so

$$Q = \frac{|Q|}{|m|} m = \frac{2022}{\sqrt{18}} (-4, 1, 1) = \left(\frac{-8088}{\sqrt{18}}, \frac{2022}{\sqrt{18}}, \frac{2022}{\sqrt{18}} \right)$$