Name——————————————, **ID** ————————

Final Exam , MTH 205 , Summer 2021

Ayman Badawi

(Stop working at 10:am/ submit your solution by 10:15 am / DO NOT SUBMIT BY EMAIL) ———— 44

QUESTION 1. (**4 points**)(**SHOW THE WORK**)

Find $y(t)$, where $y' - 2y = U_2(t) - \delta_2(t), y(0) = 0$

QUESTION 2. (**SHOW THE WORK**)(**4 points**)

Solve for $x(t)$ and $y(t)$, where $x(0) = y(0) = 0$ and

$$x(t) + y'(t) = t^2 + 2t$$
$$x'(t) - y(t) = 0$$

QUESTION 3. (**SHOW THE WORK**)(**4 points**) Find $y(t)$, where

$$(t+1)y^{(2)} + y' = 1, y(0) = 1, y'(0) = 0, \ \ t > -1$$

.

QUESTION 4. (**SHOW THE WORK**)(**4 points**) Solve the D.E. (You might need $\int (f(x)+f'(x))e^x dx = f(x)e^x + c$):

$$\frac{dy}{dt} = \frac{1}{sin(y) + cos(y) - t}$$

QUESTION 5. (**SHOW THE WORK**)(**4 points**) Solve the D.E.

$$(\frac{x}{y} + 1)dx + (1 - \frac{x^2}{y^2})dy = 0$$

QUESTION 6. (**SHOW THE WORK**)(**4 points**) Solve the D.E.

$$\frac{dy}{dx} = \frac{(2x + y + 1)^2 + 5}{2x + y - 1}$$

QUESTION 7. (**SHOW THE WORK**)(**4 points**) Consider the autonomous D.E:

$$y'(x) = y^3 - 4y^2 + 4y.$$

(1) Find all critical values and classify each as stable, unstable, or semistable.

(2) If $(-4, 1)$ lies on the graph of the solution, then Sketch such graph.

QUESTION 8. (**SHOW THE WORK**)(**4 points**) Find $y(t)$ if $y_1(t) = \frac{1}{\sqrt{t}}$ is a solution to the D.E:

$$ty^{(2)} - (1+t)y' + a_0(t)y = 0, \quad t > 0$$

You do not need to find $a_0(t)$.

QUESTION 9. (**SHOW THE WORK**)(**6 points**) Find y_g (the general solution) for the D.E :

$$ty^{(2)} + y' - \frac{y}{t} = 4ln(t), \quad t > 0$$

QUESTION 10. (**SHOW THE WORK**)(**6 points**) Imagine an object weighing 128 pounds is attached to a spring. The spring stretches 2 ft. The object starts moving with an upward initial velocity 2ft/min (i.e., $x'(0) = -2$) by displacing it 0.5 ft below the equilibrium position (i.e., $x(0) = 0.5$). Assume there is no air resistance and no external force. Note that gravity $g = 32$.

1)Find the equation of motion, $x(t)$.

2) Find the phase angle ϕ and rewrite $x(t)$ in terms of ϕ.

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Final Exam – Hafsah Tahir ①

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Q1

$$y' - 2y = u_2 - \delta_2 \qquad y(0) = 0$$

Aply laplace:

$$sY(s) - \cancel{y(0)}^{0} - 2Y(s) = \frac{e^{-2s}}{s} - e^{-2s}$$

$$Y(s)\,[s-2] = \frac{e^{-2s}}{s} - e^{-2s}$$

$$Y(s) = e^{-2s}\left[\frac{1}{s(s-2)} - \frac{1}{s-2}\right]$$

↳ using cover method: $\frac{A}{s} + \frac{B}{s-2} = -\frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s-2}$

$$Y(s) = e^{-2s}\left[-\frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s-2} - \frac{1}{s-2}\right]$$

$$= e^{-2s}\left[-\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s-2}\right]$$

laplace inverse:

$$\boxed{y(t) = \left(-\frac{1}{2} - \frac{1}{2}e^{2(t-2)}\right)u_2}$$

4

Q2

Q2

$$x(t) + y'(t) = t^2 + 2t$$

$$x'(t) - y(t) = 0$$

My fault

Apply laplace:

$$X(s) + sY(s) - \cancel{y(0)} = \frac{2}{s^3} + \frac{2}{s^2}$$

$$sX(s) - x(0) - Y(s) = 0$$

Simplify:

$$X(s) + sY(s) = \frac{2+2s}{s^3}$$

$$sX(s) - Y(s) = 0$$

Determinate:

$$X(s) = \frac{\begin{vmatrix} \frac{2+2s}{s^3} & s \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & s \\ s & -1 \end{vmatrix}} = \frac{-\frac{2+2s}{s^3}}{-1-s^2} = \frac{2+2s}{s^3(1+s^2)} = X(s)$$

4

Partial fraction:

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+1}$$

$$As^2(s^2+1) + Bs(s^2+1) + C(s^2+1) + Ds^4 + Es^3 = 2+2s$$

$A+D=0$ $\boxed{D=2}$ $A+C=0$ $\boxed{A=-2}$ $B+E=0$ $\boxed{E=-2}$ $\boxed{B=2}$ $\boxed{C=2}$

$$X(s) = -\frac{2}{s} + \frac{2}{s^2} + \frac{2}{s^3} + \frac{2s-2}{s^2+1}$$

$\mathcal{L}^{-1}$ → $\boxed{x(t) = -2 + 2t + t^2 + 2\cos(t) - 2\sin(t)}$

$x' = y(t)$ → $\boxed{y(t) = 2 + 2t - 2\sin(t) - 2\cos(t)}$

Q3

$(t+1)y'' + y' = 1$

$\omega = y'$
$\omega' = y''$

$(t+1)\omega' + \omega = 1$ first order LDE

1. $\omega' + \frac{1}{t+1}\omega = \frac{1}{t+1}$

2. $I = e^{\int \frac{1}{t+1} dt} = \cancel{e}^{\cancel{\ln}(t+1)} = t+1$

3. $\int \cancel{t+1} \cdot \frac{1}{\cancel{t+1}} dt = \frac{t+c}{t+1} = \omega = y'$

$$\begin{array}{r} 1 \\ t+1 \overline{)\, t} \\ -t-1 \\ \hline -1 \end{array}$$

$\int \frac{t+c}{t+1} dt = \int 1 - \frac{1}{t+1} + \frac{c}{t+1} dt$

$= t - \ln(t+1) + c\ln(t+1) + k = y$

$y(0) = 1 = -\ln(1) + c\ln(1) + k \rightarrow k = 1$

$y'(0) = 0 = \frac{c}{1} = c = 0$

$\boxed{y = t - \ln(t+1) + 1}$ $t > -1$

4

Q4

$$\frac{dy}{d\cancel{x}t} = \frac{1}{\sin(y)+\cos(y)-t}$$

$$\int(\sin y + \cos y - t)\,dy = \int dt$$

$$-\cos y + \sin y - ty = t + c$$

$$\boxed{\sin y - \cos y - ty - t = c}$$

not partial

flip

A

1st order

$$t' + t = \sin y + \cos y$$

$$t = (\sin y) + Ce^{-y}$$

Q5

$$\left(\frac{x}{y}+1\right)dx + \left(1-\frac{x^2}{y^2}\right)dy = 0$$

α - homogeneous degree 0   sub in $y = ux$, $u = \frac{y}{x}$

$$\left[f_x(1,u) + u f_y(1,u)\right]dx + x f_y(1,u)\, d\tfrac{y}{x} = 0$$

$$\left[\cancel{\tfrac{1}{u}} + 1 + u - \cancel{\tfrac{1}{u}}\right]dx + x\left(1-\frac{1}{u^2}\right)du = 0$$

$$(1+u)\,dx + x\left(\frac{u^2-1}{u^2}\right)du = 0$$

$$\int \frac{1}{x}dx + \int \frac{\frac{u^2-1}{u^2}}{1+u}\,du = 0$$

$$\int \frac{1}{x}dx + \int \frac{(u-1)\cancel{(u+1)}}{u^2\cancel{(u+1)}}\,du = 0$$

$$\int \frac{1}{x}dx + \int \frac{1}{u} - u^{-2}\,du = 0$$

$$\ln(x) + \ln(u) + \frac{1}{u} = C$$

$$\boxed{\ln x + \ln\left(\frac{y}{x}\right) + \frac{x}{y} = C}$$

✓ 4

Q6

$$\frac{dy}{dx} = \frac{(2x+y+1)^2+5}{2x+y-1}$$

$$= \frac{(2x+y+1)^2+5}{2x+y+1-1-1} = \frac{(2x+y+1)^2+5}{2x+y+1-2}$$

$u = 2x+y+1$

$\frac{du}{dx} = 2 + \frac{dy}{dx}$

$$\frac{du}{dx} - 2 = \frac{u^2+5}{u-2}$$

$$\frac{du}{dx} = \frac{u^2+5}{u-2} + 2 \rightarrow \frac{du}{dx} = \frac{u^2+5+2u-4}{u-2} = \frac{u^2+2u+1}{u-2}$$

$$\frac{du}{dx} = \frac{(u+1)^2}{u-2}$$

$$\int \underbrace{\frac{u-2}{(u+1)^2}}\, du = \int dx$$

$\hookrightarrow$ partial fractions $= \frac{A}{u+1} + \frac{B}{(u+1)^2} \rightarrow Au + A + B = u - 2$

$A = 1 \quad B = -3$

$$\int \frac{1}{u+1} - \frac{3}{(u+1)^2}\, du$$

let $v = u+1$
$dv = du$

$\int \frac{1}{v} - 3v^{-2}\, dv$

$\ln|v| + 3v^{-1}$

$$\ln(u+1) + \frac{3}{u+1} = x + c$$

$$\ln(2x+y+2) + \frac{3}{2x+y+2} = x + c$$

$$\boxed{\ln(2x+y+2) + \frac{3}{2x+y+2} - x = c}$$

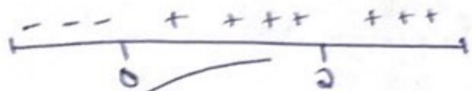
4

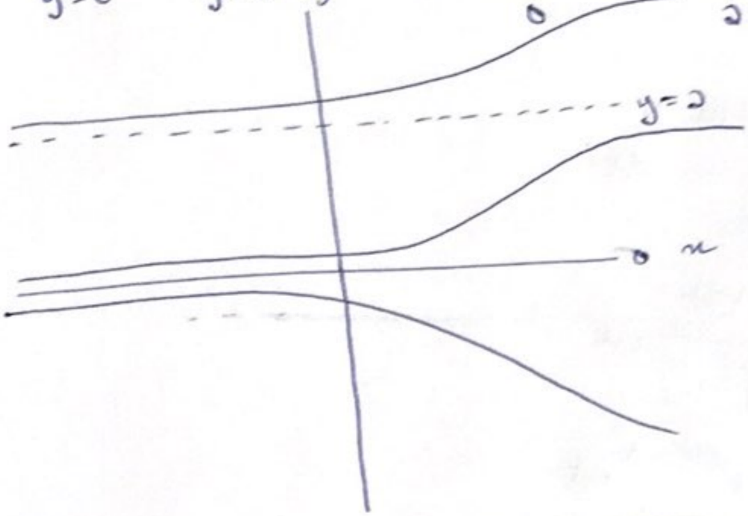
Q7

① $y^3-4y^2+4y=0$

$y(y^2-4y+4)=0$

$y=0$ $\quad$ $y=2$

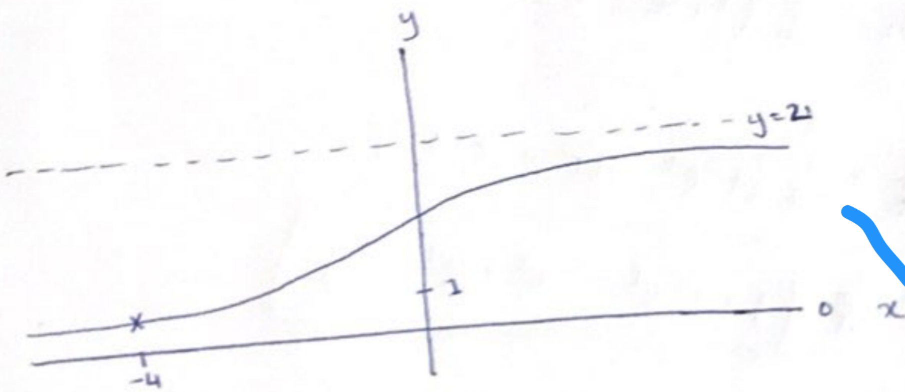




Critical points ⇒ $\boxed{\begin{array}{l} y=0 \; : \text{unstable} \\ y=2 \; : \text{semi stable} \end{array}}$

4

②



Q8 8

$ty'' - (1+t)y' + a_0(t)y = 0$

$$y_2 = y_1 \int \frac{e^{-\int \frac{a_1}{a_2} dt}}{y_1^2} dt$$

$$y_2 = \frac{1}{\sqrt{t}} \int \frac{e^{-\int -\frac{(1+t)}{t} dt}}{\frac{1}{t}} dt$$

$$y_2 = \frac{1}{\sqrt{t}} \int \frac{e^{\int \frac{1}{t} + 1 dt}}{\frac{1}{t}} dt$$

$$y_2 = \frac{1}{\sqrt{t}} \int t \cdot \cancel{e}^{\ln(t)+t} dt$$

$$= \frac{1}{\sqrt{t}} \int t^2 e^t dt$$

$'$	$\int$
t^2	e^t
$2t$	e^t
2	e^t
0	e^t

$$y_2 = \frac{1}{\sqrt{t}} (\cancel{t}\, t^2 e^t - 2te^t + 2e^t + c)$$

$$y(t) = \frac{1}{\sqrt{t}} + \frac{1}{\sqrt{t}} (t^2 e^t - 2te^t + 2e^t + c)$$

$$\boxed{y(t) = \frac{1}{\sqrt{t}} (1 + t^2 e^t - 2te^t + 2e^t + c)}$$

4

Q9

$$t y'' + y' - \frac{y}{t} = 4\ln(t)$$

$y = t^m$

$m^2 \cancel{- m + m} - 1 = 0$

$m = \pm 1 \rightarrow y_h = C_1 t + C_2 t^{-1}$

$y_p \Rightarrow$ variation

$V_1' t + V_2' t^{-1} = 0$

$V_1' \cancel{t} - V_2' t^{-2} = \frac{4\ln(t)}{t}$

$$\begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix} \quad -t^{-1} - t^{-1} = -2t^{-1}$$

$$\begin{vmatrix} 0 & t^{-1} \\ \frac{4\ln t}{t} & -t^{-2} \end{vmatrix} \quad \frac{\cancel{-} \frac{\cancel{4}^{2}\ln t}{t^{\cancel{2}}}}{\cancel{-}\cancel{2} \cdot \frac{1}{\cancel{t}}} = \frac{2\ln(t)}{t} = V_1'$$

$V_1 = \int \frac{2\ln(t)}{t} dt$

$u = \ln t$
$du = \frac{1}{t} dt$

$= 2\int u \, du$

$= u^2$

$V_1 = \ln^2(t)$

$\rightarrow$

$$\begin{vmatrix} t & 0 \\ 1 & \frac{4\ln t}{t} \end{vmatrix} \quad \frac{4\ln t}{\frac{-2}{t}} = -2t\ln(t) = v_2'$$

$$v_2 = \int -2t\ln(t)\,dt$$

$$u = \ln t \qquad v = -t^2$$
$$du = \frac{1}{t}dt \qquad dv = -2t$$

$$-t^2\ln t + \int t\,dt$$

$$v_2 = -t^2\ln(t) + \frac{1}{2}t^2$$

$$y_p = t(\ln^2(t)) + t^{-1}\left(-t^2(\ln(t)) + \frac{1}{2}t^2\right)$$

$$y_p = t\ln^2(t) + \frac{1}{2}t - t\ln(t)$$

$$\boxed{y_g = C_1 t + C_2 t^{-1} + t\ln^2(t) + \frac{1}{2}t - t\ln(t)}$$

6

Q10

$\omega = 128$

$m = \frac{128}{32} = 4 = m$

$\frac{128}{2} = 64 = k$

$V_0 = 0$

$f(x) = 0$

① $mx'' + V_0 x' + kx = f(x)$

$4x'' + 64x = 0$

$x'' + 16x = 0$

$y = e^{mt}$

$m^2 + 16 = 0$

$m = \pm 4i \rightarrow C_1 \cos 4t + C_2 \sin 4t = x(t)$

$x(0) = \boxed{C_1 = 0.5}$

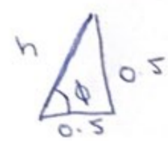
$x' = -4C_1 \sin 4t + 4C_2 \cos 4t$

$x'(0) = 4C_2 = -2$

$\boxed{C_2 = -0.5}$

$\boxed{0.5\cos 4t - 0.5 \sin 4t = x(t)}$

② $\sqrt{0.5^2 + (-0.5)^2} = \frac{\sqrt{2}}{2} = h$



$\sin\phi = \frac{o}{h}$

$0.5 = \frac{o}{h}$

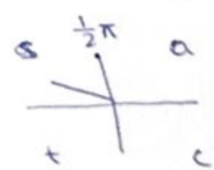
$\cos\phi = \frac{a}{h}$

$-0.5 = \frac{a}{h}$

sin = +ve

$\frac{0.5}{(\sqrt{2}/2)} = \frac{\sqrt{2}}{2}$

$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{4}\pi + \frac{1}{2}\pi = \boxed{\frac{3}{4}\pi = \phi}$



$\boxed{x(t) = \sin\left(\frac{3}{4}\pi + 4t\right)}$

6