

Exam Three, MTH 205 , Summer 2021

Ayman Badawi

(Stop working at 14:45 pm/submit your solution by 15:00 pm / DO NOT SUBMIT BY EMAIL) 46**QUESTION 1. (8 points)(SHOW THE WORK)**

A metal bar at temperature of 100C is placed in a room with constant temperature of 22c. After 20 minutes the temperature of the bar is 60C.

- (i) Find the time it will take the bar to reach a temperature of 30C.(Give your answer to the nearest one decimal)
- (ii) Find the temperature of the bar after 15 minutes. (Give your answer to the nearest one decimal)

QUESTION 2. (SHOW THE WORK)(8 points) A 50-gallons tank initially contains 10 gallons of fresh water (i.e., at $t = 0$, amount of salt is zero). A brine solution containing one pound of salt per gallon is poured into the tank at the rate of 4 gal/min, while the well-stirred mixture leaves the tank at the rate of 2 gal/min.

- (i) Find the amount of salt in the tank after 10 minutes.
- (ii) Find the concentration of the salt in the tank after 10 minutes.
- (iii) When will an overflow occur?

QUESTION 3. (SHOW THE WORK)(6 points) Solve the following D.E.

$$\frac{dy}{dx} = \frac{1}{x - x^2y^2}$$

QUESTION 4. (SHOW THE WORK)(6 points) Solve the following D.E. where $t > 0$

$$\frac{y'}{t^2} + 3y = \left(3 + \frac{1}{t^2}\right)e^t$$

QUESTION 5. (SHOW THE WORK)(6 points) Solve the following D.E.

$$(xy + y^2)dx + (x^2 - xy)dy = 0$$

QUESTION 6. (SHOW THE WORK)(6 points) Solve the following D.E.

$$(2xy + y^2 + e^x + \cos(x) + 1)dx + (x^2 + 2yx + 3y^2 + 7\sin(y) + 1)dy = 0$$

QUESTION 7. (SHOW THE WORK)(6 points) Solve the following D.E. [Try substitution then separable]

$$\frac{dy}{dx} = \frac{2x(2x + y)^8}{\sqrt{1 + x^2}} - 2$$

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Q1

$$t=0 \quad T=100, \quad T_m=22, \quad T_{20}=60$$

$$T' = k(T - T_m)$$

$$T' = kT - kT_m$$

$$\textcircled{1} T' - kT = -kT_m$$

$$\textcircled{2} I = e^{\int -k dt} = e^{-kt}$$

$$\textcircled{3} \int e^{-kt} \cdot -kT_m dt = \frac{T_m e^{-kt} + c}{-k} = T_m + ce^{kt} = T$$

$$T = 22 + ce^{kt}$$

$$100 = 22 + c \rightarrow c = 78 \rightarrow T = 22 + 78e^{kt}$$

$$60 = 22 + 78e^{20k} \rightarrow 38 = 78e^{20k}$$

$$\frac{38}{78} = e^{20k} \rightarrow \ln\left|\frac{19}{39}\right| = 20k \rightarrow \frac{1}{20} \ln\left|\frac{19}{39}\right| = k$$

$$\rightarrow T = 22 + 78e^{\left(\frac{1}{20} \ln\left(\frac{19}{39}\right)t\right)}$$

$$\textcircled{i} \quad 30 = 22 + 78e^{\left(\frac{1}{20} \ln\left(\frac{19}{39}\right)t\right)}$$

$$\frac{8}{78} = e^{\left(\frac{1}{20} \ln\left(\frac{19}{39}\right)t\right)}$$

$$\ln\left(\frac{4}{39}\right) = \frac{1}{20} \ln\left(\frac{19}{39}\right)t$$

$$\frac{20 \ln\left(\frac{4}{39}\right)}{\ln\left(\frac{19}{39}\right)} = \boxed{t = 63.3 \text{ min}}$$

$$\textcircled{\text{ii}} \quad T = 22 + 78 e^{(\frac{1}{70} \ln(\frac{19}{24}) \cdot 15)}$$

$$T = 67.5^\circ\text{C}$$

Q2

$$A'(t) = \text{In} - \text{Out}$$

$$\text{In} = 1 \times 4 = 4$$

$$\text{concentration out} = \frac{A}{10 + (4-2)t} = \frac{A}{10+2t}$$

$$\text{Out} = \frac{A}{10+2t} \times 2 = \frac{A}{5+t}$$

$$A' = 4 - \frac{A}{5+t}$$

$$\textcircled{1} \quad A' + \frac{A}{5+t} = 4$$

$$\textcircled{2} \quad I = e^{\int \frac{1}{5+t} dt} = e^{\ln(5+t)} = 5+t$$

$$\textcircled{3} \quad \int 4 \cdot (5+t) dt = \int 20 + 4t dt = \frac{20t + 2t^2 + c}{5+t} = A(t)$$

~~scribble~~

$$\begin{array}{r}
 2t+10 \\
 t+5 \overline{) 2t+20t} \\
 \underline{-2t+10t} \\
 10t \\
 \underline{-10t-50} \\
 -50
 \end{array}
 \rightarrow 2t+10 + \frac{c-50}{5+t} = A(t)$$

$$\text{at } t=0 \rightarrow 10 + \frac{c-50}{5} = 0$$

$$c-50 = -50$$

$$c=0$$

(next page)

$$\rightarrow 2t + 10 - \frac{50}{5+t} = A(t)$$

③

$$\textcircled{i} \quad t=10 \rightarrow 2(10) + 10 - \frac{50}{15}$$

$$\checkmark \quad 30 - \frac{50}{15} = \boxed{26.6 \text{ pounds of salt}} = A(t)$$

$$\textcircled{ii} \quad \frac{26.6}{10 + 2(10)} = \frac{26.6}{30} = \boxed{0.89 \text{ pounds per gallon}}$$

iii

$$\frac{A}{10+2t}$$

$$\rightarrow 10+2t=50$$

$$2t=40$$

$$t=20 \text{ min}$$



Q3

4

$$\frac{dy}{dx} = \frac{1}{x - x^2 y^2}$$

flip

$$\frac{dx}{dy} = x - x^2 y^2$$

$$x' - x = -x^2 y^2 \quad (\text{Bernoulli - non linear})$$

$$w = x^{-1}$$

$$\textcircled{1} w' + w = -y^2$$

$$\textcircled{2} I = e^{\int 1 dy} = e^y$$

$$\textcircled{3} \int e^y \cdot -y^2 dy = \frac{-y^2 e^y - 2y e^y + 2e^y + c}{e^y}$$

$$= -y^2 - 2y + 2 + c e^{-y} = x^{-1}$$

$$\text{or } \boxed{\frac{1}{y^2 - 2y + 2 + c e^{-y}} = x}$$

	\int
$-y^2$	$-e^y$
$-2y$	$-2e^y$
2	$2e^y$
0	$c e^y$

Q4

5

$$\frac{y'}{t^2} + 3y = \left(3 + \frac{1}{t}\right) e^t$$

first order LDE

$$\textcircled{1} y' + 3t^2 y = (3t^2 + 1) e^t$$

$$\textcircled{2} I = e^{\int 3t^2 dt} = e^{t^3}$$

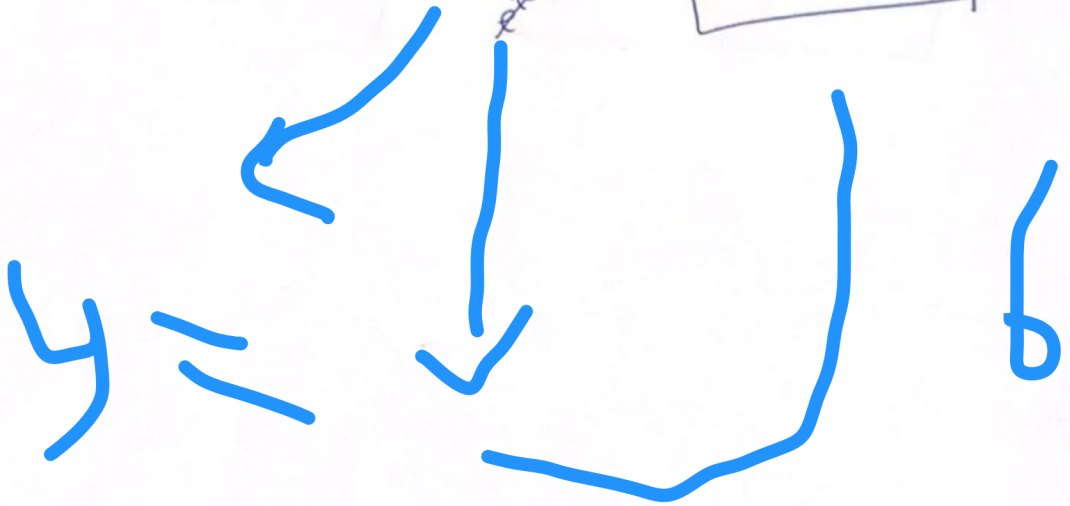
$$\textcircled{3} \int e^{t^3} \cdot e^t (3t^2 + 1) dt = \cancel{e^{t^3} (3t^2 + 1) dt}$$

$$= \int e^{t^3+t} (3t^2 + 1) dt$$

$$\text{let } u = t^3 + t \\ du = 3t^2 + 1 dt$$

$$= \int e^u du$$

$$= e^u + c = \frac{e^{t^3+t} + c}{e^{t^3}} = \boxed{e^t + ce^{-t^3} = y}$$



Q5

6

$$(xy+y^2)dx + (x^2 - xy)dy = 0$$

α -homogeneous degree 2

$$\rightarrow y = ux \\ u = \frac{y}{x}$$

$$[f_x(1,u) + u f_y(1,u)]dx + \int x f_y(1,u) du = 0$$

$$[u + u^2 + u - u^2]dx + x(1-u)du = 0$$

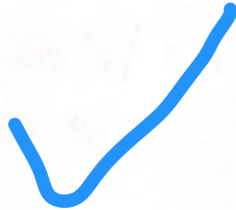
$$(2u)dx + x(1-u)du = 0$$

$$\int \frac{1}{x} dx + \int \frac{1-u}{2u} du$$

$$\ln|x| + \int \frac{1}{2u} - \frac{1}{2} du = 0$$

$$\ln(x) + \frac{1}{2} \ln(u) - \frac{1}{2} u = C$$

$$\boxed{\ln(x) + \frac{1}{2} \ln\left(\frac{y}{x}\right) - \frac{1}{2} \frac{y}{x} = C}$$



Q61

7

$$(2xy + y^2 + e^x + \cos(x) + 1) dx + (x^2 + 2yz + 3y^2 + 7\sin(y) + 1) dy = 0$$

$$f_{xy} = 2x + 2y \quad f_{yx} = 2x + 2y \quad \rightarrow \underline{\text{exact}}$$

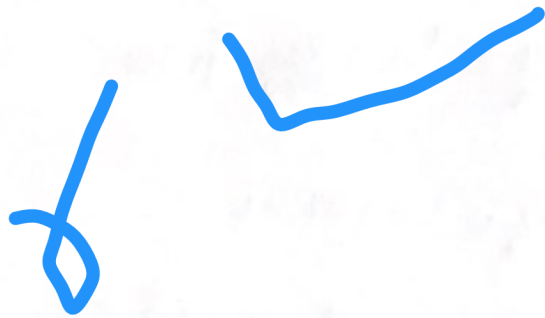
$$\int (2xy + y^2 + e^x + \cos(x) + 1) dx$$

$$x^2y + y^2x + e^x + \sin(x) + x + h(y)$$

$$f_y = x^2 + 2yx + h'(y) = x^2 + 2yx + 3y^2 + 7\sin(y) + 1$$

$$h(y) = y^3 - 7\cos(y) + y$$

$$\rightarrow x^2y + xy^2 + e^x + \sin(x) + x + y^3 - 7\cos(y) + y = C$$



Q7

$$\frac{dy}{dx} = \frac{2x(2x+y)^8}{(1+x^2)^{1/2}} - 2$$

$$\text{let } u = 2x+y$$

$$\frac{du}{dx} = 2 + \frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2$$

$$\frac{dy}{dx} \rightarrow 2 = \frac{2x}{\sqrt{1+x^2}} \cdot u^8 \rightarrow 2$$

$$\int u^{-8} du = \int \frac{2x}{\sqrt{1+x^2}} dx$$

$$\text{let } v = x^2+1 \\ dv = 2x dx$$

$$-\frac{1}{7} u^{-7} = \int \frac{2x}{\sqrt{1+x^2}} dx = \int v^{-1/2} dv$$

$$-\frac{1}{7} u^{-7} = 2v^{1/2} + c$$

$$-\frac{1}{7} (2x+y)^{-7} = 2\sqrt{x^2+1} + c$$

$$\boxed{-\frac{1}{7(2x+y)^7} = 2\sqrt{x^2+1} + c}$$

