Name——————————————, **ID** ——————

Exam Three, MTH 205 , Summer 2021

Ayman Badawi

(Stop working at 14:45 pm/ submit your solution by 15:00 pm / DO NOT SUBMIT BY EMAIL) ——— 46

QUESTION 1. (**8 points**)(**SHOW THE WORK**)

A metal bar at temperature of 100C is placed in a room with constant temperature of 22c. After 20 minutes the temperature of the bar is 60C.

(i) Find the time it will take the bar to reach a temperature of 30C.(Give your answer to the nearest one decimal)

(ii) Find the temperature of the bar after 15 minutes. (Give your answer to the nearest one decimal)

QUESTION 2. (**SHOW THE WORK**)(**8 points**) A 50-gallons tank initially contains 10 gallons of fresh water (i.e., at t = 0, amount of salt is zero). A brine solution containing one pound of salt per gallon is poured into the tank at the rate of 4 gal/min, while the well-stirred mixture leaves the tank at the rate of 2 gal/min.

(i) Find the amount of salt in the tank after 10 minutes.

(ii) Find the concentration of the salt in the tank after 10 minutes.

(iii) When will an overflow occur?

QUESTION 3. (**SHOW THE WORK**)(**6 points**) Solve the following D.E.

$$\frac{dy}{dx} = \frac{1}{x - x^2y^2}$$

QUESTION 4. (**SHOW THE WORK**)(**6 points**) Solve the following D.E. where $t > 0$

$$\frac{y'}{t^2} + 3y = (3 + \frac{1}{t^2})e^t$$

QUESTION 5. (**SHOW THE WORK**)(**6 points**) Solve the following D.E.

$$(xy + y^2)dx + (x^2 - xy)dy = 0$$

QUESTION 6. (**SHOW THE WORK**)(**6 points**) Solve the following D.E.

$$(2xy + y^2 + e^x + cos(x) + 1)dx + (x^2 + 2yx + 3y^2 + 7sin(y) + 1)dy = 0$$

QUESTION 7. (**SHOW THE WORK**)(**6 points**) Solve the following D.E. [**Try substitution then separable**]

$$\frac{dy}{dx} = \frac{2x(2x + y)^8}{\sqrt{1 + x^2}} - 2$$

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Exam 3 - Hafsah Tahir

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①

Q1

$t=0 \quad T=100 \; , \quad T_m = 22 \; , \quad T_{20} = 60$

$T' = k(T - T_m)$

$T' = kT - kT_m$

① $T' - kT = -kT_m$

② $I = e^{\int -k\,dt} = e^{-kt}$

③ $\int e^{-kt} \cdot -kT_m \, dt = \dfrac{T_m \cancel{e^{-kt}} + c}{\cancel{e^{-kt}}} = \underline{T_m + ce^{kt} = T}$

$T = 22 + ce^{kt}$

$100 = 22 + c \rightarrow \underline{c = 78} \rightarrow T = 22 + 78e^{kt}$

$60 = 22 + 78e^{20k} \rightarrow 38 = 78e^{20k}$

$\frac{38}{78} = e^{20k} \rightarrow \ln\left|\frac{19}{39}\right| = 20k \rightarrow \underline{\frac{1}{20}\ln\left|\frac{19}{39}\right| = k}$

$\rightarrow \underline{T = 22 + 78e^{\left(\frac{1}{20}\ln\left(\frac{19}{39}\right)t\right)}}$

(i) $30 = 22 + 78e^{\left(\frac{1}{20}\ln\left(\frac{19}{39}\right)t\right)}$

$\frac{8}{78} = e^{\left(\frac{1}{20}\ln\left(\frac{19}{39}\right)t\right)}$

$\ln\left(\frac{4}{39}\right) = \frac{1}{20}\ln\left(\frac{19}{39}\right)t$

$\dfrac{20\ln\left(\frac{4}{39}\right)}{\ln\left(\frac{19}{39}\right)} = \boxed{t = 63.3\text{min}}$

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(ii) $T = 22 + 78\, e^{\left(\frac{1}{20}\ln\left(\frac{19}{39}\right)\cdot 15\right)}$ ②

$\boxed{T = 67.5\,°C}$

Q2

$A'(t) = \text{In} - \text{Out}$

$\text{In} = 1 \times 4 = 4$

concentration out $= \frac{A}{10+(4-2)t} = \frac{A}{10+2t}$

$\text{Out} = \frac{A}{\cancel{10} + \cancel{2}t} \times \cancel{2} = \frac{A}{5+t}$

$A' = 4 - \frac{A}{5+t}$

① $A' + \frac{A}{5+t} = 4$

② $I = e^{\int \frac{1}{5+t} dt} = \cancel{e}^{\cancel{\ln}(5+t)} = 5+t$

③ $\int 4\cdot(5+t)\,dt = \int 20 + 4t\,dt = \frac{20t + 2t^2 + c}{5+t} = A(t)$

$$
\begin{array}{r}
2t + 10 \\
t+5 \,\big)\, \overline{\cancel{2t^2} + 20t} \\
\underline{-\cancel{2t^2} \mp 10t} \\
+\cancel{10t} \\
\underline{-\cancel{10t} - 50} \\
-50
\end{array}
\quad \rightarrow \quad 2t + 10 + \frac{c-50}{5+t} = A(t)
$$

at $t=0 \rightarrow \quad 10 + \frac{c-50}{5} = 0$

$c - \cancel{50} = \cancel{-50}$

$c = 0$

(next page)

$\rightarrow 2t + 10 - \frac{50}{5+t} = A(t)$

(i) $t = 10 \rightarrow 2(10) + 10 - \frac{50}{15}$

$30 - \frac{50}{15} = \boxed{26.6 \text{ pounds of salt}} = A(t)$

(ii) $\frac{26.6}{10 + 2(10)} = \frac{26.6}{30} = \boxed{0.89 \text{ pounds per gallon}}$

(iii) $\frac{A}{10+2t}$

$\rightarrow 10 + 2t = 50$

$2t = 40$

$\boxed{t = 20 \text{min}}$

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Q3

$$\frac{dy}{dx} = \frac{1}{x - x^2y^2}$$

flip

$$\frac{dx}{dy} = x - x^2y^2$$

$x' - x = -x^2y^2$ (bernolli - non linear)

$w = x^{-1}$

① $w' + w = y^2$

② $I = e^{\int 1 dy} = e^y$

③ $\int e^y \cdot y^2\, dy = \dfrac{y^2\cancel{e^y} - 2y\cancel{e^y} + 2\cancel{e^y} + c}{\cancel{e^y}}$

'	$\int$
y^2	e^y
$2y$	e^y
2	e^y
0	e^y

$= y^2 - 2y + 2 + ce^{-y} = x^{-1}$

or $\boxed{\dfrac{1}{y^2 - 2y + 2 + ce^{-y}} = x}$

Q4

$$\frac{y'}{t^2} + 3y = \left(3 + \frac{1}{t^2}\right)e^t$$

first order LDE

① $y' + 3t^2 y = (3t^2+1)\,e^t$

② $I = e^{\int 3t^2 dt} = e^{t^3}$

③ $\int e^{t^3} \cdot e^t (3t^2+1)\,dt =$ [illegible]

$$= \int e^{t^3+t}(3t^2+1)\,dt$$

let $u = t^3 + t$

$du = 3t^2 + 1\,dt$

$$= \int e^u\,du$$

$$= e^u + c = \frac{e^{t^3+t} + c}{e^{t^3}} = \boxed{e^t + ce^{-t^3} = y}$$

y =

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Q5 ⑥

$(xy+y^2)\,dx + (x^2 - xy)\,dy = 0$

a-homogeneous degree 2 $\rightarrow y = ux$

$u = \frac{y}{x}$

$[f_x(1,u) + u f_y(1,u)]\,dx + x f_y(1,u)\,du = 0$

$[u + \cancel{u^2} + u - \cancel{u^2}]\,dx + x(1-u)\,du = 0$

$(2u)\,dx + x(1-u)\,du = 0$

$\int \frac{1}{x}\,dx + \int \frac{1-u}{2u}\,du$

$\ln|x| + \int \frac{1}{2u} - \frac{1}{2}\,du = 0$

$\ln(x) + \frac{1}{2}\ln(u) - \frac{1}{2}u = C$

$\boxed{\ln(x) + \frac{1}{2}\ln\left(\frac{y}{x}\right) - \frac{1}{2}\frac{y}{x} = C}$

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Q6i

$(2xy+y^2+e^x+\cos(x)+1)\,dx + (x^2+2yx+3y^2+7\sin(y)+1)\,dy=0$

$f_{xy}=2x+2y$ $\quad$ $f_{yx}=2x+2y$ $\rightarrow$ exact

$\int 2xy+y^2+e^x+\cos(x)+1\,dx$

$x^2y+y^2x+e^x+\sin(x)+x+h(y)$

$f_y = \cancel{x^2}+\cancel{2yx}+h'(y)=\cancel{x^2}+\cancel{2yx}+3y^2+7\sin(y)+1$

$h(y)=y^3-7\cos(y)+y$

$\boxed{\rightarrow x^2y+xy^2+e^x+\sin(x)+x+y^3-7\cos(y)+y=c}$

Q7 ⑧

$$\frac{dy}{dx} = \frac{2x(2x+y)^8}{(1+x^2)^{1/2}} - 2$$

let $u = 2x+y$

$$\frac{du}{dx} = 2 + \frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2$$

$$\frac{du}{dx} \cancel{-2} = \frac{2x}{\sqrt{1+x^2}} \cdot u^8 \cancel{-2}$$

$$\int u^{-8}\,du = \int \frac{2x}{\sqrt{1+x^2}}\,dx$$

let $v = x^2+1$

$dv = 2x\,dx$

$$-\frac{1}{7}u^{-7} = \int v^{-\frac{1}{2}}\,dv$$

$$-\frac{1}{7}u^{-7} = 2v^{\frac{1}{2}} + c$$

$$-\frac{1}{7}(2x+y)^{-7} = 2\sqrt{x^2+1} + c$$

$$\boxed{\frac{-1}{7(2x+y)^7} = 2\sqrt{x^2+1} + c}$$

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