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Exam I

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Consider Math as Minor or Second major

$\frac{51}{51}$

QUESTION 1. (1)(**6 points**) Let n be an odd integer. Prove directly that $7n + 6$ is an odd integer, i.e., show that $7n + 6 = 2m + 1$ for some integer m.

$n = 2k+1$, for some integer k

$7(2k+1)+6 = 2m+1$, for some integer m

$14k + 7 + 6 = 2m+1$

$14k + 12 + 1 = 2m+1$

$2(\underbrace{7k+6}_{\in \mathbb{Z}})+1 = 2m+1$

$\therefore 7n+6$ is odd when n is an odd integer

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(2)(**6 points**) Use the 4th method and prove that $\sqrt{22}$ is an irrational number.

6

deny that $\sqrt{22}$ is irrational:

$\sqrt{22} = \frac{a}{b}$, for $a, b \in \mathbb{Z}$, $b \neq 0$

$22 = \frac{a^2}{b^2}$ $\longrightarrow$ a^2 has to be even $\longrightarrow$ a has to be even

$\longrightarrow$ b^2 has to be odd $\longrightarrow$ b has to be odd

$22 = \frac{(2n)^2}{(2m+1)^2}$, for $n, m \in \mathbb{Z}$

$22(2m+1)^2 = (2n)^2$

$\frac{22(4m^2) + 22(4m) + 22}{4} = \frac{4n^2}{4}$ $\longrightarrow$ $\underbrace{22m^2 + 22m + \frac{22}{4}}_{\notin \mathbb{Z}} = \underbrace{n^2}_{\in \mathbb{Z}}$

the denial is invalid. Therefore, $\sqrt{22}$ is irrational

(3) (**3 points**) Use (2) above and prove by contradiction that $\sqrt{2} + \sqrt{11}$ is an irrational number.

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deny that $\sqrt{2} + \sqrt{11}$ is irrational:

$\sqrt{2} + \sqrt{11} = \frac{a}{b}$, for $a, b \in \mathbb{Z}$, $b \neq 0$

$(\sqrt{2} + \sqrt{11})^2 = \frac{a^2}{b^2}$

$2 + 2\sqrt{22} + 11 = \frac{a^2}{b^2}$

$2\sqrt{22} = \frac{a^2}{b^2} - 13$ $\longrightarrow$ $\underbrace{\sqrt{22}}_{\text{irrational}} = \left[\frac{\frac{a^2}{b^2} - 13}{2}\right]$ rational

$\therefore$ the denial is invalid. Therefore, $\sqrt{2} + \sqrt{11}$ is irrational

(4) **(3 points)** Find all values of x in the PLANET Z_{15} that satisfy the equation $\overset{a}{9}x = \overset{b}{6}$ in Z_{15}.

$gcd(9,15) = 3$

is $3 \mid 6$? yes

$\therefore$ 3 solutions exist

3

$x_1 = 4$ ⟶ $x_2 = 4 + \frac{15}{3} = 9$

$x_3 = 9 + 5 = 14$

Solution set: $\{4, 9, 14\}$

(5) **(3 points)** Find $7^{1600002}$ $mod(40)$

$40 = 20 \times 2$

10×2^2

5×2^3

3

$\phi(40) = (5^0)(4) \times (2^2)(1)$

$= 16$ ⟶ $1600002 \pmod{16} = 2$

$7^{1600002} \pmod{40} = 7^2 \pmod{40}$

$= 9$

QUESTION 2. (6 points) Let x be the number of females in MTH 213. Given $1 \le x < 88$ such that $x\ (mod\ 11) = 8$ and $x\ (mod\ 8) = 3$. Find the value of x.

$m_1 = 11,\ m_2 = 8$

$r_1 = 8,\ r_2 = 3$

$gcd(11, 8) = 1$

$\therefore$ CRT can be used

$m = 11 \times 8 = 88$

$n_1 = \frac{88}{11} = 8$

$n_2 = \frac{88}{8} = 11$

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n_1^{-1} in $Z_{m_1} \longrightarrow (8)^{-1}$ in $Z_{11} = 7$

n_2^{-1} in $Z_{m_2} \longrightarrow (11)^{-1}$ in $Z_8 = 3$

$x = [(8)(7)(8) + (11)(3)(3)] \pmod{88}$

$= 547 \pmod{88}$

$= 19$ ⟵ smallest positive x

other x's can be generated from this formula: $19 + 88k$

[illegible]

QUESTION 3. (6 points)

Use Math Induction and prove that $\sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} = \frac{n}{3n+9}$ for every $n \geq 1$. [Hint: note 3n + 9 = 3(n + 3)]

1) prove $\sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} = \frac{n}{3n+9}$ for the smallest n (n = 1)

left hand side: $\sum_{i=1}^{1} \frac{1}{(i+2)(i+3)} = \frac{1}{(1+2)(1+3)} = \frac{1}{12}$

right hand side: $\frac{1}{3(1)+9} = \frac{1}{12}$

2) assume that $\sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} = \frac{n}{3n+9}$ is true for some $n \geq 1$

3) prove that $\sum_{i=1}^{n+1} \frac{1}{(i+2)(i+3)} = \frac{n+1}{3(n+1)+9} \left(= \frac{n+1}{3n+12} \right.$

$$\sum_{i=1}^{n+1} \frac{1}{(i+2)(i+3)} = \sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} + \frac{1}{((n+1)+2)((n+1)+3)}$$

$$= \frac{n}{3n+9} + \frac{1}{(n+3)(n+4)}$$

$$= \frac{n}{3(n+3)} + \frac{1}{(n+3)(n+4)}$$

$$= \frac{n(n+4) + 3}{3(n+3)(n+4)}$$

$$= \frac{n^2+4n+3}{3(n+3)(n+4)}$$

$$= \frac{(n+1)\cancel{(n+3)}}{3\cancel{(n+3)}(n+4)}$$

$$= \frac{n+1}{3(n+4)}$$

$$= \frac{n+1}{3n+12}$$

∴ therefore, $\sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} = \frac{n}{3n+9}$ for every $n \geq 1$ as proven by induction

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QUESTION 4. (5 points) True or False

(i) $\forall x \in Z^*, \exists y! \in Q$ such that $xy = 2$. true

(ii) If $\exists x \in Z$ such that $x^2 - 3 = 0$, then now you are having a cup of coffee on the moon. true

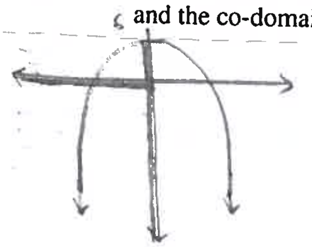
(iii) $x^2 + 5x + 6 = 0$ for some $x \in N$ if and only if $y^2 - 8 = 0$ for some $y \in Z$ true

(iv) $\exists\, x \in Z$ such that $\forall y \in R$, $yx^2 - 4y = 0$. true

$y(x^2 - 4) = 0$

(v) $\exists!\, y \in R$ such that $\forall x \in Z^*$, $|xy| = 3|x|$ false

QUESTION 5. (5 points) Let $f : (-\infty, 0] \to (-\infty, 6]$ such that $f(x) = 6 - x^2$. If f is invertible, find the domain and the co-domain of f^{-1}, then find the equation of f^{-1}.



by horizontal line test, we can see that any horizontal line would intersect f at exactly 1 point

∴ f is both one-to-one and onto, therefore, f^{-1} exists.

$f^{-1}: (-\infty, 6] \to (-\infty, 0]$

$x = 6 - y^2$

$-y^2 = x - 6$

$y^2 = 6 - x$

$y = -\sqrt{6-x}$

since codomain is $(-\infty, 0]$

QUESTION 6. (4 points) Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 5 & 2 & 4 & 3 \end{pmatrix}$

Find the smallest positive integer n such that $f^n = I$.

disjoint cycles: $\underbrace{(1\ 7\ 3)}_{\text{3-cycle}}\underbrace{(2\ 6\ 4\ 5)}_{\text{4 cycle}}$

$n = LCM[3,5] = \frac{3 \times 5}{gcd(3,5)} = 15$

QUESTION 7. (4 points) Let $A = \{3, 1, 5, 7\}$ and $B = \{3, 2, 9\}$ Write True or False

(i) $\{(3,7), (9,1)\} \in P(B \times A)$ true

(ii) $\{\{3\}, \{9\}\} \subseteq P(B)$ true

(iii) $\{1, 7\} \subseteq P(A)$ false

(iv) $\{(5,3)\} \in A \times B$ false