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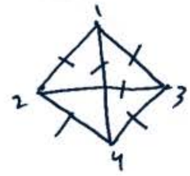
MTH 418. Graph Theory. Final Exam

Ayman Badawi

QUESTION 1. (1) **(6 points)** decompose K_4 into copies of P_4.

K_4 can be decomposed into 2 copies of P_4



$P_4 : 1-2-3-4$ $\qquad K_4 = P_4 \oplus P_4$

$P_4 : 2-4-1-3$

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(2) **(6 points)** Let $G(V, E)$ be a finite graph such that the degree of at least 11 vertices is an odd integer. Prove that G can not be decomposed into 5 paths.

in a path, we have ~~eat~~ exactly 2 vertices w/ odd degrees, and interior vertices have even degree

Hence, in 5 paths, we have 10 vertices w/ odd degree

↳ ∴ the graph G needs to have 10 vertices w/ odd degree

but, since G has >10 vertices w/ odd degree, it cannot be decomposed into 5 paths.

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(3) **(6 points)** Let G be a 4-regular finite graph (i.e., degree(each vertex) = 4) with girth not equal to 3. Prove that G must have at least 8 vertices. $\boxed{\text{Construct such graph with 8 vertices.}}$ → H

let $x, y \in V$ since girth $\neq 3$, then if $x - y \in E$ $\qquad |V| \geq |N(x)| + |N(y)|$

$$N(x) \cap N(y) = \phi$$

$$|N(x)| \wedge |N(y)| = \underbrace{4 + 4}_{\text{since deg(each vertex) = 4.}} = 8$$

∴ G must have at least 8 vertices. $\qquad |V| \geq 8$

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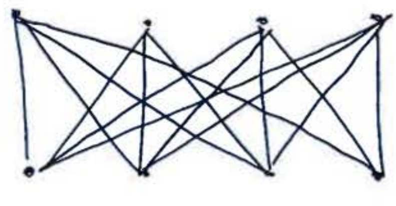

H is $K_{4,4}$.

(4) **(6 points**) Let G be a graph of order 2024 and size 2022. Prove that G is never connected.

~~ont~~ we know that every connected graph has a spanning subgraph that is a tree, and a tree is the smallest connected graph.
we also know that a tree has order n and size $n-1$
since G has size $= 2022 < 2024 - 1$, G cannot be connected

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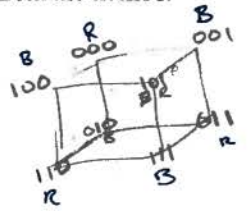
(5) **(10 points**) Is Q_3 a bipartite graph? construct the largest cycle in Q_3. Give me a minimum dominating set of Q_3. Give me a maximum matching set? Give me a maximum independent set? What is the chromatic number of Q_3? What is the edge-chromatic number of Q_3?



1) largest cycle in Q_3 is $2^3 = 8 \Rightarrow C_8$

000 – 001 – 101 – 100 – 110 – 111 – 011 – 010

2) min. dominating set : $\gamma = 2$
$= \{000, 111\}$

3) max. matching set $= \{000-100,\ 001-101,\ 010-110,\ 111-011\}$.
we have perfect matching set here ☺

4) max. indep. set $= \{000, 101, 110, 011\}$.

5) chromatic no : 2

6) edge chromatic no : 3

7) is Q_3 bipartite? yes. we can split the vertices in 2 groups s.t.
$A = \{$vertices w/ odd parity$\}$
$B = \{$vertices w/ even parity$\}$.

since we know that we can only have odd – even – odd – even (2 even vertices cannot be adjacent, and 2 odd vertices can't be adjacent)

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(6) (**3 points**) Given C_m is an induced subgraph of K_{11}. Find all possible values of m.

$m = 3$

3

(7) (**3 points**) For what values of m is $K_{8,m}$ a Hamiltonian path?

~~[illegible]~~ $m = 7, 8, \text{ or } 9$

3

(8) (**3 points**) Prove that $K_{3,3}$ is not a planar.

$K_{3,3} \to$ no C_3 cycle

$|V| = 6$

$|E| = \frac{3(6)}{2} = 9$

$|E| \leq 2|V| - 4$

$9 \leq 2(6) - 4$

$9 \leq 8 \Rightarrow$ contradiction

$\therefore K_{3,3}$ is not planar.

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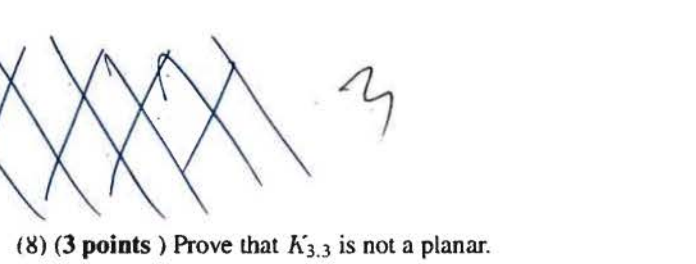
(9) (**3 points**) Let G be a connected graph such that at least one vertex is of degree 3. Prove that the line graph (L(G)) is never a bipartite graph.

3 M:

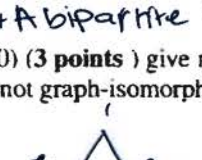


if this graph, M, is a subgraph of G, then we will ~~[illegible]~~ always end up with a C_3 in the line graph of G

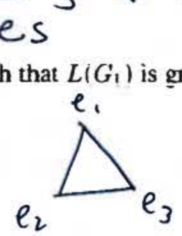
we know that A bipartite graph has no odd cycles $\therefore$ L(G) is never bipartite.

(10) (**3 points**) give me an example of two graphs G_1, G_2 such that $L(G_1)$ is graph-isomorphic to $L(G_2)$, but G_1 is not graph-isomorphic to G_2.

G_1:



$L(G_1)$:



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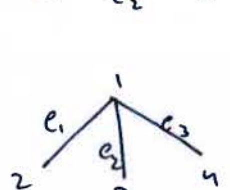
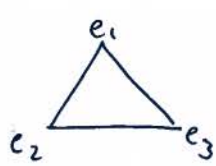
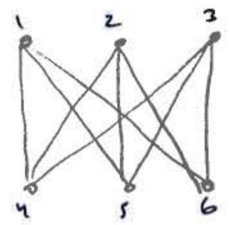
G_2:



$L(G_2)$:



(11) **(3 points)** Consider the graph $K_{3,3}$. If it has a vertex-cut, then name all of them.



$K_{3,3}$ does not have a vertex cut
removing a vertex, say vertex 1 (has max. deg)
will remove 1-4, 1-5, 1-6 but the graph
is still connected. the same applies to all other
vertices.

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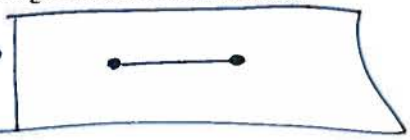
(12) **(3 points)** Give me an example of a connected graph that has an edge-cut but not a vertex cut.



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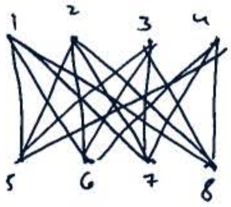
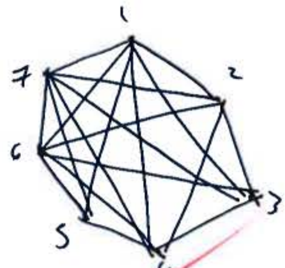
(13) **(3 points)** Give me a minimum vertex-cover set of $K_{4,4}$ and a minimum vertex-cover set of K_7.



min. vertex cover set $(K_{4,4}) = \{1, 2, 3, 4\}$.



1/3 min. vertex ~~cover~~ set $(K_7) = \{1, 2, 3, 4, 5, 6\}$.
(this is the min; if we remove any more vertices, we will lose $V_6 - 7$ edges).

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirate
E-mail: abadawi@aus.edu, www.ayman-badawi.com