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MTH 418, Graph Theory, Exam Two

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$deg(v) \geq 3$

QUESTION 1. (6 points) Let $G(V, E)$ be a graph such that $degree(v) \geq 2024$ for every $v \in V$. Prove that G must have an EVEN cycle.

$deg(v) \geq 2024$ implies $deg(v) \geq 3$ $\forall v \in V$.

Let P be the longest path in G s.t $P: v_1 - v_2 - \cdots - v_m$

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Since $deg(v) \geq 2024$, we know v_m must be connected to atleast 2 more vertices $v_i, v_k \neq v_{m-1}$. ($v_i, v_k \in P$ since, otherwise, we'd construct a path longer than P, a contradiction). So we have:

$$v_1 - v_2 - \cdots - v_i - \cdots - v_k - \cdots - v_{m-1}$$ with v_m joined to v_i and v_k.

1) If the length of the path $v_i - \cdots - v_m$ is odd, then $v_i - \cdots v_m - v_i$ is an even cycle.

2) If the length of the path $v_i - \cdots - v_m$ is even then:
(a) lengths of $v_i - \cdots - v_k$ and $v_k - \cdots - v_m$ are both even OR
(b) length of $v_i - \cdots - v_k$ is odd and $v_k - \cdots - v_m$ is odd,

Then for case (a), $v_i - \cdots - v_k - v_m - v_i$ forms an even cycle.
for case (b), $v_k - \cdots - v_m - v_k$ is an even cycle.

Hence, G has an even cycle

QUESTION 2. (6 points) Let $G(V, E)$ be a connected graph such that P, Q are paths of maximum length. Prove that P, Q must have at least one vertex in common.

Ans 2: Let $P_1: v_1 - v_2 - \cdots v_m$ and $P_2: w_1 - w_2 - \cdots - w_m$ be such 2 paths of max. length m and assume they do not intersect. Since G is connected, we must have a path P that connects v_1 to w_1. (see picture):

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$$v_1 - v_2 - \cdots - v_i - \cdots - v_m$$
$$\Big\} Q$$
$$w_1 - w_2 - \cdots - w_i - \cdots - w_m$$

Then $P: v_1 - v_2 - \cdots - v_i - q_1 - \cdots - q_k - w_i - \cdots - w_2 - w_1$. Note that v_i can $= v_1$ or w_i can $= w_1$.

Let the length of $v_1 - v_2 - \cdots - v_i$ be n and the length of $w_1 - w_2 - \cdots - w_i$ be L. Case 1: If $n + L \geq m$, then the length of P is $n + L$ + length of $Q > m$ ↯. Case 2: If $n + L < m$, then we have the path $P_y: \underbrace{v_m - v_{m-1} - \cdots - v_i}_{m-n} - \underbrace{q_1 - \cdots - q_k}_{} - \underbrace{w_i - \cdots w_m}_{m-L}$ of length

$m - n +$ length of $Q + m - L = 2m - (L+n) +$ length $Q > m +$ length Q since $L + n < m$. ↯ Hence P_1, P_2 must intersect

QUESTION 3. (a) (**6 points**) Let G be a tree of order $n \geq 2$. Prove that G has size equals to (n-1). [Hint: if you need the result that says if the degree of every vertex of a connected graph is ≥ 2, then G must have a cycle, just say: We know that if the degree of every vertex of a connected graph is ≥ 2, then G must have a cycle

Proof by induction:

(1) Base case: $n=2$. $G: v_1 - v_2$. By staring, $|E| = |V| - 1 = 2-1 = 1$.

(2) Assume that a tree $G(V,E)$ of order n has size $n-1$. We show that a tree $G_1(V_1, E_1)$ of order $n+1$ has size n. Assume that $\forall v \in V_1$, $deg(v) \geq 2$, then we know that G_1 must have a cycle. ↯ since G_1 is a tree. Hence, $\exists\, a \in V_1$ s.t $deg(a) = 1$ and $\exists!\, b \in V_1$ s.t $a-b \in E_1$. Consider $G = G_1(V_1 - \{a\}, E_1 - \{a-b\})$. Since G_1 has no cycles and $deg(a)=1$ hence removing it does not disconnect the graph, then G is a tree of order n. By the hypothesis, size of $G = |E_1 - \{a-b\}| = n-1$.
$\Rightarrow |E_1| = n-1+1 = n$.

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(b) (**3 points**) Let G be a graph of order 7 and size 5. Prove that G is never a connected graph.

We know that for any connected graph, ~~the~~ $\exists$ a spanning subgraph that is a tree. Since the tree is spanning G, then it is of order 7, size 6 (by (a)). By removing an edge from the tree, we disconnect the graph. Hence, any connected graph of order 7 has size ≥ 6.
$\therefore G$ of order 7, size 5 cannot be connected.

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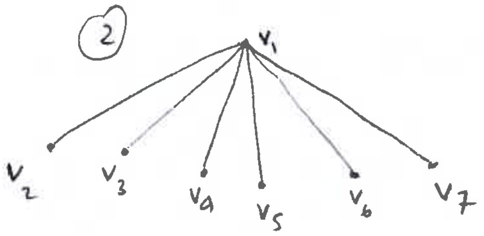
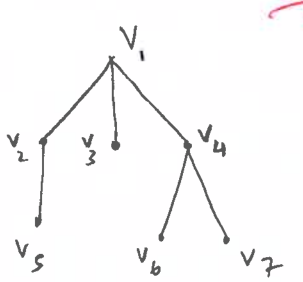
(c) (**3 points**) Draw three non-isomorphic trees, each is of order 7.

(1) $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7$

(2)



(3)



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QUESTION 4. Consider Q_4, the 4-dimensional hypercube.

(i) **(6 points** Is Q_4 a bipartite-graph? If no, explain briefly. If yes, then find A, B, $|A|$, and $|B|$.

Yes, Q_n is bipartite $\forall n \geq 1$. Since by construction, if $v_1 - v_2 \in E$, the parity of v_1 and v_2 are different Then let $A = \{v \in V \mid v$ has even parity$\}$ and let $B = \{v \in V \mid v$ has odd parity$\}$. Q_4 has $2^4 = 16$ vertices s.t $v_1 - v_2 - \dots - v_{16}$ is a subgraph of Q_4. Assume that v_1 has odd parity, then v_2 has even parity, v_3 has odd parity .... Hence $\forall$ odd $1 \leq i \leq 16$, v_i has odd parity. Then $A = \{v_1, v_3, v_5, v_7, v_9, v_{11}, v_{13}, v_{15}\}$. $|A| = 8$
$B = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}, v_{16}\}$. $|B| = 8$

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(ii) **(3 points**) How many edges does Q_4 have?

$|E| = \dfrac{4 \cdot 2^4}{2} = 32$ ✓

(iii) **(3 points)** Let P be a path of maximum length in Q_4. Construct P.

P: 0000 – 0001 – 0011 – 0010 – 0110 – 0100 – 1100 – 1000 – 1001 – 1011 – 1010 – 1110 – 1111 – 0111 – 0101 – 1101

$\Longrightarrow P_{16} \to$ note that C_{2^4} is the largest cycle a subgraph of Q_4

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(iv) **(3 points)** Find $d(0011, 1101) = 3$

3

(v) **(3 points)** Find $diam(Q_4) = d(0000, 1111) = 4$

3

QUESTION 5. (~~3~~ 6 **points)** Let G be a graph that is not connected. Assume that G has exactly 2 vertices, v, w, of odd degrees. Prove that there is a path between v, w.

Let $C_1, C_2, \dots, C_k$ be components of G s.t $G = C_1 \cup C_2 \cup \dots \cup C_k$. Assume $\nexists$ path between v and w. Then $v \in C_i$, $w \in C_j$ $1 \leq i, j \leq k$ and $i \neq j$. # edges in $C_i = \dfrac{\sum \text{degrees}}{2}$. Since all vertices $\in C_i$ except v have even degree, then $\sum$ degree $= 2k + 1$. $k \in \mathbb{Z}^+$.

$\Rightarrow |E_i| = \dfrac{2k+1}{2} \neq$ integer. $\lightning$. Hence, $\exists$ a path between v and w.

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QUESTION 6. (a)(**3 points**) Given C_m, $m \geq 3$ is an induced subgraph of K_{2024}. Find all possible values of m.

$m = 3$

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(b) (**3 points**) Given C_i, $i \geq 3$, is an induced subgraph of $K_{n,m}$. Find all possible values of i

$i = 4$

3

$\exists\, v \in V$ s.t. $\forall w \in V$, $v \in N(w)$

$\boxed{\text{Deny. Hence } G \text{ has neither } C_4 \text{ nor } P_4 \text{ as an induced subgraph.}}$ Thus by a class result

(c) (**3 points**) Let G be a connected graph such that for every $v \in V$, there is a vertex $w \in V$ such that $w \notin N(v)$. Prove that G must have an induced C_4 subgraph or an induced P_4. [Hint: Think about a result in the class notes!] Assume not. Since G is connected, we must have a path P from w to v, $|P| \geq 2$. Let P be such shortest path ($|P| = 2$). Then P: $w - z - v$. Let $x \in N(v)$. Then we have P_L: $w - z - v - x$. If $z - x \notin E$ and $w - x \notin E$, then P_L is an induced P_4. If $z - x \notin E$, and $w - x \in E$, then we have $w - z - v - x - w$, induced C_4.

v, w, x, z

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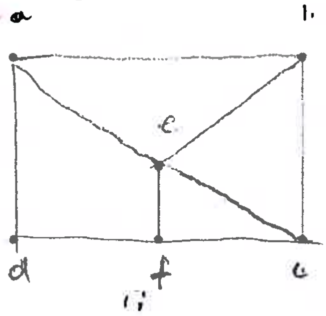
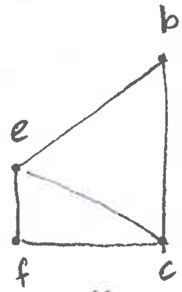
Long

not clear

(d) (**3 points**) Stare at the subgraph H of G. Is H an induced subgraph of G? explain briefly.





H

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Yes, $\forall\, v, w \in H$, if $v - w \in E_G$, then $v - w \in E_H$

(e) (**3 points**) Find the adjacency matrix of H.

$$\begin{array}{c|c} & \begin{array}{cccc} e & f & b & c \end{array} \\ \begin{array}{c} e \\ f \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

3 ✓

(f) (**6 points**) Is the sequence $3, 3, 2, 1, 1, 1, 1$ graphical? If yes, draw such graph

3 3 2 1 1 1 1

2 1 0 1 1 1

$\Rightarrow$ 2 1 1 1 1 0

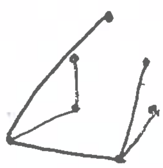
0 0 1 1 0

$\Rightarrow$ 1 1 0 0 0. Yes it is graphical.

6 ✓

$w - z - v - x$

$w - z - x$, v



Faculty information

1 3 1 2 3 1 1

3 3 2 1 1 1 1 ✓

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