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$\frac{51}{51}$

MTH 418, Graph Theory, Exam One

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$V_1 -$, $V_1 - V_2 - V_3 - V_{n}$

QUESTION 1. (**6 points**) Let $G(V, E)$ be a finite graph such that $degree(v) \geq 2$ for every $v \in V$. Prove that G must have a cycle.

Let P_n be the longest path on G s.t $P_n = v_1 - v_2 - \cdots - v_n$. Since the degree of every vertex ≥ 2, $deg(v_1)$ and $deg(v_n)$ must be atleast 2. This means that v_1 must be connected to a vertex $w \neq v_2$. If $w \notin \{v_3, \ldots, v_n\}$, then we'll be constructing a path $w - v_1 - v_2 - \cdots - v_n$ longer than the longest path P_n $\lightning$. $\therefore$ $w \in \{v_3, \ldots, v_n\}$. The same argument applies for v_n. This means that we will have the cycle $v_1 - v_2 - \cdots - w - v_1$ and the cycle $v_n - v_{n-1} - \cdots - w - v_n$.

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QUESTION 2. (**6 points**) Let $G(V, E)$ be a finite graph such that the degree of at least 9 vertices is an odd integer. Prove that G can not be decomposed into 4 paths.

In any path, the degree of the internal vertices is even (2) whereas the degree of the endpoints is odd (1). For any vertex to have an odd degree, it must appear as an endpoint on a path an odd number of times. In 4 paths, we can only have a maximum of 8 distinct endpoints with odd degrees. $\therefore$ We cannot decompose 9 vertices of odd degrees into 4 paths.

6

QUESTION 3. (i) (**3 points**)Can we decompose K_{15} into two isomorphic subgraphs? explain briefly.

We know / K_n can be decomposed onto 2 isomorphic subgraphs iff $4 \mid n$ or $4 \mid n-1$.

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Since $4 \nmid 15$ and $4 \nmid 14$, K_{15} cannot be decomposed onto 2 isomorphic subgraphs.

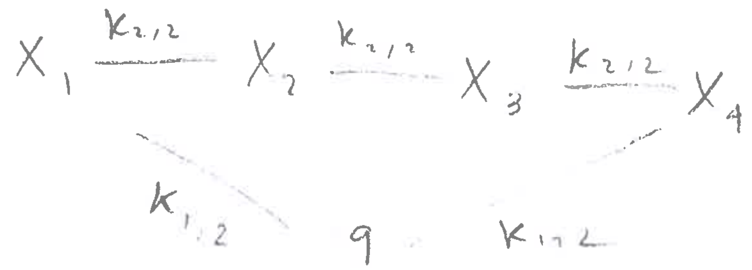
(ii) (**6 points**) As explained in the class, construct the subgraphs H, K such that $K_9 = H \oplus K$, where H is graph-isomorphic to K. Note there are other decomposition of K_9, but this decomposition satisfies $\overline{H} = K$ and $\overline{K} = H$. So, add this piece of information to your dictionary. You can check that after the exam, do not show it now.

$K_9(V,E)$. Let $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

let $X_1 = \{1, 2\}$ $X_2 = \{3, 4\}$ $X_3 = \{5, 6\}$ $X_4 = \{7, 8\}$

Then:

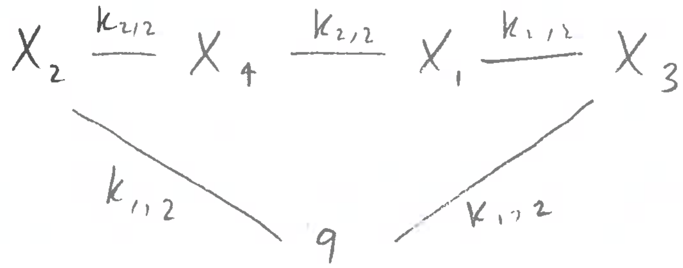
H :



s.t $X_1 = K_2$ and $X_4 = K_2$

and

K :



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s.t $X_2 = K_2$ and $X_3 = K_2$

QUESTION 4. Let $G(V, E)$ be the Petersen graph as constructed in the class. 1, 2, 3, 4, 5

12
13
14
15
23
24
25
34
35
45

(i) **(3 points)** Find $N(23)$, as explained in the class, note that the vertex 23 is the set $\{2, 3\} = \{3, 2\}$ and it is the same vertex 32.

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$N(23) = \{14, 15, 45\}$

(ii) **(3 points)** Find the girth of the Petersen graph (do not construct the cycle)

3

girth = 5

(iii) **(6 points)** I claim that C_8 is a subgraph of the Petersen graph, i.e., the Petersen graph has a cycle of length 8. If you believe me, construct such cycle. If not, explain briefly. C_8

Yes:

C_8 : 12 — 35 — 14 — 23 — 45 — 13 — 25 — 34 — 12

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QUESTION 5. (6 points) Let G be a 21-regular finite graph (i.e., degree(each vertex) = 21) with girth not equal to 3. Prove that G must have at least 42 vertices. Construct such graph with 42 vertices. $G(V, E)$

Let $v_1 - v_2 \in E$. Since girth$(G) \neq 3$, $N(v_1) \cap N(v_2) = \phi$

And since $N(v_1) \cap N(v_2) = \phi$ and $\deg(v_1) = \deg(v_2) = 21$

Then, $|N(v_1)| = 21$ + $|N(v_2)| = 21$ distinct vertices

we have atleast

$\therefore$ We have atleast 42 distinct vertices.

$\rightarrow$ such graph is $K_{21,21}$.

6

12 — 35 — 14 — 23 — 45 — 13 — 25 — 34 — 12

QUESTION 6. (1) (**6 points**) decompose K_6 into copies of P_6. 3 copies

P_6 : 1 — 2 — 3 — 4 — 5 — 6

P_6 : 2 — 4 — 6 — 1 — 5 — 3

P_6 : 4 — 1 — 3 — 6 — 2 — 5

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$K_6 = P_6 \oplus P_6 \oplus P_6$

(2) (**6 points**) We know that $K_{6,2}$ can be decomposed into copies of C_4, C_6, or C_8. Explicitly, write down all possible decomposition.

max length = 2(2) = 4

$K_{6,2} = C_4 \oplus C_4 \oplus C_4$ (12 edges).

C_4 : 1 — 3 — 2 — 4 — 1

C_4 : 1 — 5 — 2 — 6 — 1

C_4 : 1 — 7 — 2 — 8 — 1

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