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Final Exam

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$\frac{43}{48}$

QUESTION 1. .

(i) **(6 points)** Up to isomorphism, classify all non-cyclic abelian groups of order $125 = 5^3$

$125 = 5^3$ (Prime Factorization)

	Possible Groups
5^3	$\mathbb{Z}_{125}$
$5^2 \times 5$	$\mathbb{Z}_{25} \oplus \mathbb{Z}_5$ ✓
$5 \times 5 \times 5$	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$ ✓

We want non-cyclic so $(\mathbb{Z}_{125}, +)$ is eliminated

Therefore all non-cyclic Abelian groups of order 125 are isomorphic to either $(\mathbb{Z}_{25}, +) \oplus (\mathbb{Z}_5, +)$ or $(\mathbb{Z}_5,+) \oplus (\mathbb{Z}_5,+) \oplus (\mathbb{Z}_5,+)$

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(ii) **(4 points)** Let $(D, *)$ be a cyclic group of order 24. Assume $a \in D$ and $|a| = 24$. Find all subgroups of D with 8 elements. Write such subgroups in terms of a.

An element of order 8 will generate a subgroup of 8 elements, say H

$|a^k| = \frac{|a|}{\gcd(k, |a|)} = \frac{24}{\gcd(k, 24)} = 8 \Rightarrow \gcd(k, 24) = 3 \Rightarrow k = 3$ is an option

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so $H = \langle a^3 \rangle = \{a^3, a^6, a^9, a^{12}, a^{15}, a^{18}, a^{21}, a^{24} = e\}$

Because D is cyclic, $\exists!$ subgroup of 8 elements and that is H

(iii) **(4 points)** Let $(F, *)$ be a group with p^2 elements, for some prime integer p. Assume that $|C(G)| > 1$. Prove that G is an abelian group.

$C(F) < F$ so $|C(F)| \mid |F| \Rightarrow |C(F)| \mid p^2$

$\Rightarrow |C(F)| = \cancel{1}$ or $\overset{(1)}{p}$ or $\overset{(2)}{p^2}$

given $|C(F)| > 1$

(1) If $|C(F)| = p$, then $\frac{|F|}{|C(F)|} = p$ and so $\frac{F}{C(F)}$ is a group of prime order $\Rightarrow \frac{F}{C(F)}$ is cyclic

$\frac{F}{C(F)}$ is a group as $C(F) \trianglelefteq F$

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and if $\frac{F}{C(F)}$ is cyclic, then F is Abelian!

(2) If $|C(F)| = p^2$, then $\frac{|F|}{|C(F)|} = 1$ so $|C(F)| = |F|$ and so $C(F) = F$

but that means all elements in F commute with each other as all elements $\in C(F)$ $\Rightarrow F$ is Abelian!

Proved for (1) & (2)

(iv) (6 points) Let $(D,*)$ be a group with 55 elements. Assume that H, F are normal subgroups of D such that $|H| = 11$ and $|F| = 5$. Prove that D is a cyclic group.

H & F are both cyclic as their orders are prime

and $|H*F| = \frac{|H|\cdot|F|}{|H\cap F|} = \frac{11\cdot 5}{1} = 55$

$H\cap F < H$ & $H \cap F < F$ so $\underbrace{|H\cap F| \mid |H|}_{|H\cap F| \mid 11}$ & $\underbrace{|H\cap F| \mid |F|}_{|H\cap F| \mid 5}$

so $|H\cap F| = 1 = \gcd(5,11)$

$\Rightarrow H\cap F = \{e\}$

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1) $H*F = D$
2) $H \triangleleft D, F \triangleleft D$
3) $H\cap F = \{e\}$

so $D \approx \underset{\text{cyclic}}{H} \oplus \underset{\text{cyclic}}{F}$ & $\gcd(|H|,|F|) = \gcd(5,11) = 1$ so D is also cyclic $D \approx (\mathbb{Z}_{55},+)$

(v) (6 points) Let $(D,*)$ be an abelian group with 36 elements. Assume that if $a \in D$ and $a \neq e$, then $|a| \neq 4$. Up to isomorphisms, classify all such groups.

no elements of order 4

$36 = 6^2 = 2^2\cdot 3^2$

2^2	3^2	possible groups
2^2	3^2	$\mathbb{Z}_{2^2\cdot 3^2}$ X a cyclic group of 36 elements is guaranteed to have a subgroup of order 4 $\Rightarrow \exists\, a$ s.t $\lvert a\rvert = 4$
2×2	3×3	$\mathbb{Z}_{2^2}\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3$ X $\mathbb{Z}_{2^2}\oplus\{0\}\oplus\{0\}$ has an element of order 4 $(1,0,0)$
		$\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{3^2}$ ✓ $\text{lcm}(2,2,9) = \text{lcm}(2,9) = 18$ and $4 \nmid 18$ ✓
		$\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3$ ✓ $\text{lcm}(2,2,3,3) = \text{lcm}(2,3) = 6$ and $4\nmid 6$ ✓

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(vi) (6 points) Let $f : (Z_9,+) \to (Z_{13}^*,\cdot)$ be a group homomorphism such that $f(1) \neq 1$. Find $Range(f)$ and $Ker(f)$.

$|f(a)| \mid |\mathbb{Z}_{13}^*|$ & $|f(a)| \mid |\mathbb{Z}_9|$ $\forall\, a\in\mathbb{Z}_9$

$\Rightarrow |f(a)| \mid 12$ & $|f(a)| \mid 9$

12: 1, ~~2~~, 3, ~~4~~, ~~6~~, ~~12~~
9: 1, 3, ~~9~~

$\Rightarrow |f(a)| \mid 3$

$\Rightarrow |f(a)| = 1$ or $|f(a)| = 3$

but $f(1) \neq 1$ so f is not the trivial homomorphism from $(\mathbb{Z}_9,+)$ to $(\mathbb{Z}_{13}^*,\cdot)$

so $|Range(f)| = 3$ as $Range(f) < (\mathbb{Z}_{13}^*,\cdot)$

$\underset{\mathbb{Z}_9}{G}/Ker(f) \approx \underbrace{Range(f)}_{\text{only 1 subgroup of order 3 in } (\mathbb{Z}_{13}^*,\cdot) \text{ cyclic}}$

$|Ker(f)| = \frac{|G|}{|Range(f)|} = \frac{9}{3} = 3$

Only 1 subgroup of order 3 in $(\mathbb{Z}_9,+)$

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so $Range(f) = \langle 3\rangle_{(\mathbb{Z}_{13}^*,\cdot)} = \{1,3,9\}$ ✓

$Ker(f) = \langle 3\rangle_{(\mathbb{Z}_9,+)} = \{0,3,6\}$ ✓

(vii) (6 points) Let $f = (1\,4\,5)\,O\,(4\,7\,5\,8) \in S_8$.

Is $f \in A_8$? explain

$$f = (1\,4\,5)(4\,7\,5\,8) = \begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 4&2&3&5&1&6&7&8 \end{pmatrix}\begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 1&2&3&7&8&6&5&4 \end{pmatrix}$$

$$= \begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 4&2&3&7&8&6&1&5 \end{pmatrix}$$

Find $|f|$.

$$= (1\,4\,7)(5\,8) = \underbrace{(1\,7)(1\,4)(5\,8)}_{\text{odd \# of 2-cycles}}$$

so $f \notin A_8$

$|f| = \mathrm{lcm}(|(1\,4\,7)|, |(5\,8)|)$

$= \mathrm{lcm}(3, 2) = 6$

✓ 6

(viii) (6 points) Does S_{10} have an abelian subgroup of order 30? If no, explain. If yes, construct such subgroup.

We can construct an Abelian subgroup of order 30 if we can make an external product of 2 disjoint cycles s.t. the product of their orders is 30

30: 1, 2, 3, 5, 6, 10, 15, 30

30 = 1 · 30 X 30 > 10

= 2 · 15 X 15 > 10

= 3 · 10 X } overlapping elements 3+10 > 10

= 5 · 6 X } 5+6 > 10

It is not possible to make an external product of 2 disjoint sets s.t the product of their orders is 30

No it is not possible to construct such a subgroup

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(ix) (4 points) If I told you that (R, +, .) is a ring? What does that mean? If I told you that (R, +, .) is a field. What does that mean?

If $\underset{\sim}{(R,+,\cdot)}$ is a ring, then 1) $(R,+)$ is an Abelian group

$(\mathbb{Z},+,\cdot)$ for example

2) $(R,\cdot)$ is a monoid

3) $\forall a,b,c \in R$, $a(b+c) = ab + ac$

$(b+c)a = ba + ca$

Faculty information

If $\underset{\sim}{(R,+,\cdot)}$ is a field, then 1) $(R,+)$ is an Abelian group

$(\mathbb{Q},+,\cdot)$ for example

Rational

2) $(R^*,\cdot)$ is an Abelian group

3)

✓