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Exam II

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$\frac{40}{43}$ Excellent

QUESTION 1. Let $G = (Z_6, +) \oplus (Z_9, +)$

(i) **(6 points)** I claim that G has three distinct subgroups of order 3. Prove it or prove me wrong. Yes!

$\gcd(6,9) = 3$. $Z_6 = \langle 1 \rangle$. $|1^k| = \frac{6}{\gcd(6,k)} = 3$, $k = 2$. So $H_1 = \{1^2, (1^2)^2, (1^2)^3 = 0\}$

1) $\{2, 4, 0\} \oplus \{0\}$. (Note, H is a subgroup and $\{0\}$ is a subgroup so H_1 is a subgroup)
$H_1 = \{(2,0), (4,0), (0,0)\}$.

Z_9: $|1^k| = \frac{9}{\gcd(9,k)} = 3$. $k = 3$. $H = \langle 1^3 \rangle = \langle 3 \rangle = \{3, 3^2, 3^3 = 0\}$

2) $\{0\} \oplus \{3, 6, 0\} = \{(0,3), (0,6), (0,0)\} = H_2$

3) $|(a,b)| = 3 = LCM[|a|, |b|] = 3$. So we have $|a| = 3$ and $|b| = 3 \Rightarrow a = 2$, $b = 3$. $\Rightarrow H_3 = \{(2,3), (2,3)^2, (2,3)^3 = (0,0)\} = \{(2,3), (4,6), (0,0)\}$

6

(ii) **(2 points)** Is G cyclic? explain briefly

No, since $\gcd(6,9) \neq 1$, $\nexists\, a \in G$ where $|a| = 6 \times 9 = 54$.

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QUESTION 2. (i) **(3 points)** Let $f : (Z_5, +) \to (Z_5^*, .)$ be a group homomorphism. Find $f(a)$ for every $a \in Z_5$. Explain briefly.

$|a|$ $\forall a \neq e$ $= 5$ and we know $|f(a)| \mid 5$. Also, by Lagrange, since $f(a) \in (Z_5^*, .)$, then $|f(a)| \mid 4$. Hence $|f(a)| \mid \gcd(4,5) \Rightarrow$ $|f(a)| = 1$. $\therefore f(a) = 1$

3

(ii) **(6 points)** Let $f : (Z_6, +) \to (Z_9, +)$ be a group homomorphism such that $f(1) \neq 0$. Find the range of f and the Kernel of f.

Since $\frac{|Z_6|}{|ker(f)|} = |Range(f)|$, and $|Range(f)| \mid |Z_9|$, then $|Range(f)| \mid \gcd(6,9) = 3$

$|Range(f)| \mid 3$. Hence $|Range(f)| = 1$ or 3. if $|Range(f)| = 1$,

$|ker(f)| = 6 \Rightarrow ker(f) = \{1,2,3,4,5,6\} \Rightarrow f(1) = 0$ ↯.

Hence, $|Range(f)| = 3$. $\therefore Range(f) = \{3, 3^2, 3^3 = 0\} = \{3, 6, 9\}$

and $|ker(f)| = 2$, $\therefore ker(f) = \{3, 3^2 = 0\} = \{3, 0\}$

6

finite cyclic

QUESTION 3. (i) **(4 points)** I claim that $G = (Z_7, +) \oplus (Z_{11}^*, .)$ is group-isomorphic to $(Z_n, +)$ for some integer n. If I am right, then find n and justify my claim. If I am wrong, then tell me why?

Since gcd(7, 10) = 1, then G is a finite cyclic group of size 70, and since $(Z_7, +)$ and $(Z_{11}^*, \cdot)$ are both cyclic,

Hence, we know that G is group isomorphic to $(Z_{70}, +)$ (n=70)

(Any finite cyclic group is isomorphic to $(Z_n, +)$)

X

(ii) **(4 points)** Let $(G, *)$ be a group and $C(G)$ be the center of G. Prove that $C(G)$ is a normal subgroup of G.

1) First we show $C(G) < G$. Let $x, y \in C(G)$, we show $x * y^{-1} \in C(G)$.

Let $a \in G$. Then, $x * a = a * x \Rightarrow a = x^{-1} * a * x$. and $y * a = a * y$

$\Rightarrow y * a = y * x^{-1} * a * x = a * y \Rightarrow (y * x^{-1}) * a = a * (y * x^{-1})$. Hence, $y * x^{-1} \in C(G)$, $C(G) < G$

2) We show $C(G) \lhd G$. Let $a \in G$, then $a * C(G) * a^{-1}$

$= a * a^{-1} * C(G) = e * C(G) = C(G)$.

$\therefore C(G) \lhd G$

X

(iii) **(4 points)** Let G be as in (ii). Assume that $G/C(G)$ is a cyclic group. Prove that G is abelian.

Since $\frac{G}{C(G)}$ is cyclic, then $\frac{G}{C(G)} = \langle a * C(G) \rangle$ for some $a \in G$.

Note that $(a * C(G))^m = \underbrace{(a * C(G)) \cdot_{C_G} (a * C(G)) \cdot_{C_G} \cdots \cdot_{C(G)} (a * C(G))}_{m \text{ times}}$

$= a^m * C(G)$. So we have $\frac{G}{C(G)} = \{ \ldots, a^{-1} * C(G), a^0 = C(G), a^1 * C(G) \ldots \}$,

Let $x, y \in G$. Then $x * C(G) = a^m * C(G)$, $y * C(G) = a^k * C(G)$. $m, k \in \mathbb{Z}$

$\Rightarrow (a^m)^{-1} * x = c_1 \in C(G)$, $(a^k)^{-1} * y = c_2 \in C(G)$.

$\Rightarrow x = a^m * c_1$, $y = a^k * c_2$. Then

$x * y = a^m * c_1 * a^k * c_2 = a^m * a^k * c_1 * c_2 = a^{m+k} * c_2 * c_1 = a^k * a^m * c_2 * c_1$

$= a^k * c_2 * a^m * c_1 = y * x$. $\therefore G$ is abelian

X ✓

(iv) **(3 points)** Is $\underbrace{(Z_4, +) \oplus (Z, +)}_{G}$ cyclic? Explain briefly.

NO. $\exists\, a \in G$, $a \neq (0,0)$ s.t $|a|$ is finite

ex: $a = (1, 0)$. $|a| = \text{LCM}(|1|, |0|) = 4 < \infty$.

But G is an infinite group. Assuming it is cyclic, we know that $\forall a \in G$, $a \neq (0,0)$ $|a| = \infty$ However, $|(1,0)| = 4 < \infty$, $(1,0) \in G$, hence, G cannot be cyclic.

3

$a * H = b * H \Rightarrow a \in b * H \Rightarrow a = b * h$.

3^3 2×3^2

QUESTION 4. (i) **(4 points)** Let G be a cyclic group with 27 elements. How many elements of order 27 does G have?

$\phi(27) = \phi(3^3) = 2\times 3^2 = 18$ elements

(ii) **(4 points)** Let $f:(G_1,*)\to(G_2,\Delta)$ be a group homomorphism. Assume that $Ker(f)=\{e_1\}$. Prove that f is One-to-One.

Let $a,b\in G_1$, s.t $f(a)=f(b)$. (we show $a=b$)

~~Since~~ Then $[f(b)]^{-1}\,\Delta\, f(a) = e_2$

$\Rightarrow f(b^{-1})\,\Delta\, f(a) = e_2$

$\Rightarrow f(b^{-1}*a) = e_2$

$\Rightarrow b^{-1}*a \in Ker(f) = \{e_1\}$

$\Rightarrow b^{-1}*a = e_1$

$\Rightarrow a=b$

Hence, f is one to one.

X

Let $a\in G$. We show $a*H=H*a$
It is clear that G has exactly 2 left cosets, $\{H, b*H\}$, for some $b\notin H$. If $a\in H$, clearly $a*H=H=H*a$.
Suppose $a\notin H$. Then $a*H=b*H$.

3) finite $a*H=H*a$ or $a*H*a^{-1}=H$ $H<G$

(iii) **(4 points)** Let G be a group with $2n$ elements for some integer n. Assume that H is a subgroup of G with exactly n elements. Prove that H is a normal subgroup of G.

$a*H = H*a$ ... $a*H$ ... H_1 another subgroup of n elements.

$x\in a*H$. $x=a*h$, show $x\in H*a$ ... $x=h_2*a$ ... $h_2\in$

$\{0,2,4,6\}$. $H_1=\{1,3,5,7\}$. $H\oplus H_1 = \mathbb{Z}_8$.

$3+H = 3$ ... $H+3$. Since abelian.

$G=H\oplus H_1$

0/2

We know $G=H\cup a*H$
$=H\cup H*a$
Since $H\cap(a*H)=\phi$ and $H\cap(H*a)=\phi$, by staring $a*H=H*a$. Thus $H\lhd G$.

$\in H$ $a*H$ $b*H$ $\notin H$.

if G is $C(G)$ cyclic, G is abelian. Hence all subgroups are normal.

Faculty information