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$S_1 \Rightarrow S_2$ 10

Exam I-MW

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$$Score = \frac{59}{59}$$

QUESTION 1. (8 points, Show the work)

(i) Let x be the number of females in MTH 213. Given $1 \le x < 40$ such that $x \pmod 7 = 3$ and $x \pmod 5 = 4$. Find the value of x.

$x \pmod{\overset{m_1}{7}} = \overset{r_1}{3}$

$x \pmod{\overset{m_2}{5}} = \overset{r_2}{4}$

$m = (7)(5) = 35$

$\Rightarrow n_1 = \frac{m_1 m_2}{m_1} = m_2 = 5 \quad \Rightarrow 5 \bmod 7 = 5, \quad 5^{-1} \text{ in } Z_7 = 3 \quad c_1 = 3$

$\Rightarrow n_2 = \frac{m_1 m_2}{m_2} = m_1 = 7 \quad \Rightarrow 7 \bmod 5 = 2 \quad 2^{-1} \text{ in } Z_5 = 3 \quad c_2 = 3$

$x = (n_1)(r_1)(c_1) + (n_2)(r_2)(c_2) \pmod m$

$x = (5)(3)(3) + (7)(4)(3) \pmod{35}$

$x = 129 \pmod{35}$

$x = 24$

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(ii) Assume that $40 < x < 80$ in (i), what is the value of x?

$7 \times 5 = 35$
$m = 35$

$24 + 35 = 59$

by staring $x = 59$

since $59 \pmod 7 = 3$ and $59 \pmod 5 = 4$

So $x = 59$

QUESTION 2. (5 points, Show the work) Use the truth table and convince me that $(S_1 + S_2) \Rightarrow S_3 = (\overline{S_1} \cdot \overline{S_2}) + S_3$

S_1	S_2	S_3	$S_1 + S_2$	$\overline{S_1} \cdot \overline{S_2}$	$(S_1 + S_2) \Rightarrow S_3$	$(\overline{S_1} \cdot \overline{S_2}) + S_3$
1	1	1	1	0	1	1
1	1	0	1	0	0	0
1	0	1	1	0	1	1
1	0	0	1	0	0	0
0	1	1	1	0	1	1
0	1	0	1	0	0	0
0	0	1	0	1	1	1
0	0	0	0	1	1	1

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By staring, we can see that $(S_1 + S_2) \Rightarrow S_3$ is $\equiv$ to $(\overline{S_1} \cdot \overline{S_2}) + S_3$. hence, we have proved it using a truth table.

QUESTION 3. **(Show the work) (1)(6 points)** Let n be an even integer. Prove directly that $8 \mid (n^2 + 2n)$, i.e., show that $n^2 + 2n = 8m$ for some integer $m \in Z$.

Since n is even, $n = 2m$ where $m \in Z$

Prove: $8 \mid (n^2 + 2n)$

$(n^2 + 2n) = ((2m)^2 + 2(2m))$
$= 4m^2 + 4m$
$= 4(m^2 + m)$ → $\in Z = x$, x is always even; since $(odd)^2 + (odd) = even$, $(even)^2 + even = even$
$= 4x = 4(2Y)$; $Y \in Z$, since x is even $x = 2Y$, $Y \in Z$
$= 8Y$

$\Rightarrow$ $8 \mid 8y$ given $y \in Z$. → $\frac{8y}{8} = y$ $y \in Z$; so $8 \mid 8y$

hence, we proved that $8 \mid (n^2 + 2n)$ given that n is an even integer.

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(2)(4 points) Use the 4th method and prove that $\sqrt{22}$ is an irrational number.

① Deny: $\sqrt{22}$ is rational

$\sqrt{22} = \frac{a}{b}$; $a \in Z$, $b \in Z$, $b \neq 0$

$22 = \frac{a^2}{b^2}$ a^2 is even; a is even $\Rightarrow a = 2m$ $m \in Z$
b^2 is odd; b is odd $\Rightarrow b = 2n+1$ $n \in Z$

$22 = \frac{(2n)^2}{(2n+1)^2}$ $\Rightarrow$ $22 = \frac{4n^2}{4n^2+4n+1}$

$22(4n^2 + 4n + 1) = 4n^2$

$(22)4n^2 + (22)4n + 22 = 4n^2$

$\underbrace{22n^2 + 22n + \frac{11}{2}}_{\text{never whole integer}} = \underbrace{m^2}_{\text{always whole integer}}$

hence; we have a condridiction. So, using the 4^{th} method we proved $\sqrt{22}$ is an irrational number.

(3) (4 points) Use (2) above and prove by contradiction that $\sqrt{2} + \sqrt{11}$ is an irrational number.

Using condridiction. Deny, $\sqrt{2} + \sqrt{11}$ is an rational numer

$\sqrt{2} + \sqrt{11} = w$ $w \in Q$

$(\sqrt{2} + \sqrt{11})^2 = w^2$

$2 + 2\sqrt{2}\sqrt{11} + 11 = w^2$

$2\sqrt{22} + 13 = w^2$

$2\sqrt{22} = \underbrace{w^2 - 13}$ → $\in Q$; as proven in class rational − rational = rational : $\frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{a_1b_2 - a_2b_1}{b_1b_2}$ $a_1, b_1, a_2, b_2 \in Z$ $b_1, b_2 \neq 0$

contridiction ← $\sqrt{22} = \underbrace{\frac{1}{2}(w^2 - 13)}$ → $\in Q$; as proven in class (rational)(rational) = rational ↳ $(\frac{a_1}{b_1})(\frac{a_2}{b_2}) = \frac{a_1a_2}{b_1b_2}$ $a_1, a_2, b_1, b_2 \in Z$ $b_1, b_2 \neq 0$

As proven in (2) $\sqrt{22}$ is irrational, so it can't equal a rational. hence, we have a condradiction. so $\sqrt{2} + \sqrt{11}$ is a irrational #

$a^{b+c} = a^b a^c$ 531441

(4) **(6 points)** Use Math Induction and prove that $7 \mid (3^{3n} + 1)$ for every ODD integer $n \geq 1$.

① prove for smallest n.

$7|(3^{3n}+1)$; $n=1$ $7|(3^3+1)$ $7|(28)$ $\frac{28}{7}=4$ proven: $7|(3^3+1)$

② Assume true for some $n \geq 1$.

$7|(3^{3n}+1)$ for some $n \geq 1$; n is odd

③ Prove true for next n.

next $n = n+2$ => since odd. => prove: $7|(3^{3(n+2)}+1)$

$7|(3^{3n+6}+1) \Rightarrow 7|(3^{3n}\cdot 3^6+1)$

$3^{3n}\cdot 3^6+1 = 3^{3n}\cdot 3^6 + 3^6 - 3^6 + 1 \Rightarrow 3^6(3^{3n}+1) + (-3^6+1)$

$\Rightarrow 7|(-3^6+1)$

$7|-728 \Rightarrow \frac{-728}{7}=-104$

Proven $7|(-3^6+1)$ ✓

From #2 $7|3^{3n}+1$

hence $7|3^6(3^{3n}+1)$

hence: $7|3^6(3^{3n}+1)+(-3^6+1)$

so: $7|(3^{3n+6}+1) \Rightarrow 7|(3^{3(n+2)}+1)$ proven ✓

Using math Induction, we proved $7|(3^{3n}+1)$ for every odd integer $n \geq 1$

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QUESTION 4. (1) **(4 points, Show the work)** Find all values of x in the PLANET Z_{15} that satisfy the equation $6x = 3$ in Z_{15}.

$ax=b$ in Z_n

$6x=3$ in Z_{15}

$15 = 2\cdot 6 + 3$, $6 = 2\cdot 3 + 0$

$\gcd(a,n) = \gcd(6,15) = 3$

is $\gcd(6,15)|15 \Rightarrow 3|15 \Rightarrow$ True ✓

There is 3 solutions, with $d = \frac{n}{\gcd(a,n)} = \frac{15}{3} = 5$

$x=3$ +5

$x=8$ +5

$x=13$

solution $= \{3, 8, 13\}$

(2) **(4 points, Show the work)** Find 3^{64004} $(mod\ 128)$

$3^{64004} \pmod{128} \Rightarrow a^{\phi(n)} \pmod n = 1$

$128 = 4\times 32 = 2\times 2\times 4\times 8 = 2\times2\times2\times2\times2\times2\times2 = 2^7$

$\phi(128) = (2^{7-1})(2-1) = (2^6)(1) = 64$

$3^{64\times 1000+4} \pmod{128}$

$3^{64\times 1000}\cdot 3^4 \pmod{128}$

$= \text{Mod}(128) = 1$ from euler's rule

$\Rightarrow 3^4 \pmod{128}$

$= 81$

$xy=1$

$y=\frac{1}{x}$

~~$2xy$~~ $2xy-3x=0$

$x(2y-3)=0$

QUESTION 5. (6 points) write True or False

(i) $\forall x \in Z^*, \exists y! \in Q^*$ such that $xy = 1$ T

$x \in Z \setminus \{0\}$ $\exists! y \in Q^*$

(ii) $\exists y \in Q^*$ such that $\forall x \in Z,\ 2xy - 3x = 0$ T

(iii) If $\underbrace{\exists x \in Q \text{ such that } x^2 - 2 = 0}_{F}$, then $\underbrace{2^2 = 2024}_{F}$ T

(iv) $\exists!\ x \in Z$ such that $\underbrace{x^2 - x - 6 = 0}_{F}$ if and only if $\underbrace{y^2 + 1 = 10}_{T}$ for some $y \in Z$ F

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QUESTION 6. (6 points) Let $A = \{3, 1, 5, 7\}$ and $B = \{3, 2, 9\}$ Write True or False

(i) $\{(3,5), (9,1)\} \in P(A \times B)$ F

(ii) $\{\phi, \{3,9\}\} \subseteq P(B)$ T

(iii) $\{(1,3)\} \subseteq P(A \times B)$ F

(iv) $\{(5,3)\} \in P(A \times B)$ T

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QUESTION 7. (6 points)

$For\ k = 2\ to\ n^2 + 1$

$x = 7 * k^3 + 3 * k^2 - 10 * k$

$For\ i = 1\ to\ k$

$y = x^4 + 2 * i^2$

$Next\ i$

$Next\ k$

a) Find the exact number of the arithmetic operations that the code will execute.

Outer loop	Inner Loop
# runs: $n^2+1-2+1=n^2$	# runs: $k-1+1=k$
# operations: 8	# operations: 6
Total: $8n^2$	Total: $6k$
	1st term = 12
	Last term = $6(n^2+1)$

$$\Rightarrow (8n^2) + n^2\left(\frac{12 + 6(n^2+1)}{2}\right)$$

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b) Find O(code)

$O(\text{code}) = n^4$