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MTH 213, Discrete Math, Fall 2024, 1–4

Exam II, TR

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$$SCORE = \frac{54}{56}$$

QUESTION 1. (i) (**4 points**) Let $A = \{3, 6, 5, 10, 40\}$ and $B = \{1, 2, 4, 8\}$ define "$\leq$" on A such that for every $a, b \in A$, "$a \leq b$" iff $b = ac$ for some $c \in B$. Then " $\leq$ " is a partial order on A. Don't show that. VIEW "$\leq$" as a subset of $A \times A$. Find all elements of "$\leq$".

$a, b \in A$, "$a \leq b$" iff $b = ac$ "$\leq$" $= \{(3,3), (6,6), (5,5), (10,10), (40,40), (3,6), (5,10), (5,40), (10,40)\}$

(ii) (**4 points**) Define "=" on $A = \{1, 4, 7, 13, 12, 2, 8, 9, 10\}$ such that for every $a, b \in A$, a" = "b iff $a \bmod(4) = b \bmod(4)$. Then " = " is an equivalence relation on A. Find all equivalence classes. If we view "=" as a subset of $A \times A$. What is the size of "=". DON'T find the elements of =.

$a \bmod 4 = b \bmod 4$

$\bar{1} = \{1, 9, 13\}$

$\bar{2} = \{2, 10\}$

$\bar{4} = \{4, 8, 12\}$

$\bar{7} = \{7\}$

are the equivalence classes

Size of $|"="| = |\bar{1}|^2 + |\bar{2}|^2 + |\bar{4}|^2 + |\bar{7}|^2$

$= 9 + 4 + 9 + 1 = 23$

(iii) (**4 points**) Convince me that $\leq = \{(1,1), (2,2), (3,3), (1,4), (3,2), (2,4)\}$ is not a partial order relation on $A = \{1, 2, 3, 4\}$. Add some pairs to $\leq$ so it becomes partial order.

there should be these elements in it.

add pairs - $\{(4,4), (3,4)\}$

since $(4,4)$ is not in $\leq$, it is not reflexive $\Rightarrow$ not partial order

since $(3,4)$ is not in $\leq$, it is not transitive as $(3,2)$ & $(2,4)$ exist.

579.

(iv) **(4 points)** Given " = " = $\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1),(2,4),(4,2),(3,5),(5,3),(1,5),(5,1)\}$ is an equivalence relation on $A = \{1,2,3,4,5\}$. Find all distinct equivalence classes of "=".

$\bar{1} = \{1,3,5\}$

$\bar{2} = \{2,4\}$

$\boxed{9+4 = 13}$

QUESTION 2. (i) **(4 points)** Let $a_n = 5a_{n-1} + 14a_{n-2}$ such that $a_1 = 10$ and $a_2 = 90$. Find a general formula for a_n.

$a_n = 5a_{n-1} + 14a_{n-2}$

$a_h = ?$, $\alpha^n - 5\alpha^{n-1} - 14\alpha^{n-2} = 0$

divide by α^{n-2},

$\alpha^2 - 5\alpha - 14 = 0$

P: -14
S: -5

$\alpha^2 - 7\alpha + 2\alpha - 14 = 0$

$\alpha_1 = 7$ & $\alpha_2 = -2$

$C_1 = \frac{110}{63}$ $C_2 = \frac{10}{9}$

$a_n = C_1\alpha_1^n + C_2\alpha_2^n$

$a_n = C_1 7^n + C_2(-2)^n$

$a_1 = C_1 7 + C_2(-2) = 10$

$= 7C_1 - 2C_2 = 10$

$a_2 = 49C_1 + 4C_2 = 90$

$a_n = C_1\alpha_1^n + C_2\alpha_2^n$

$= \left(\frac{110}{63}\right)(7)^n + \left(\frac{10}{9}\right)(-2)^n$

(ii) **(6 points)** Let $a_n = 9a_{n-1} - 18a_{n-2} + 100(2^n)$. Find a general formula for a_n. DO NOT FIND C_1, C_2.

$a_n = 9a_{n-1} - 18a_{n-2} = 0$

for general

$a_h = ?$, $\alpha^n - 9\alpha^{n-1} + 18\alpha^{n-2} = 0$

divide by α^{n-2},

$\alpha^2 - 9\alpha + 18 = 0$

P: 18
S: -9

$\alpha^2 - 6\alpha - 3\alpha + 18 = 0$

(6)(-3)

$(\alpha - 6)(\alpha - 3)$

$\alpha_2 = 6$, $\alpha_1 = 3$

$a_n = C_1\alpha_1^n + C_2\alpha_2^n$

$= C_1 3^n + C_2 6^n$

no particular but simply

$a_p = A2^n$

$A2^n - 9(A2^{n-1}) + 18A2^{n-2} = 100\, 2^n$

divide by 2^n

$A - \frac{9A}{2} + \frac{18A}{4} = 100$

$A\left(1 - \frac{9}{2} + \frac{18}{4}\right) = 100$

$\Rightarrow A = 100$

$a_n = C_1 3^n + C_2 6^n + 100\, 2^n$

0 1 2 3 4 5 6

QUESTION 3. (i) **(3 points)** The digits 0, 1, 2, ..., 6 will be used to construct car plates, where each plate has 5 digits. How many ODD number plates (i.e., last digit is ODD) can be constructed if repetition is not allowed?

$\boxed{6}\ \boxed{5}\ \boxed{4}\ \boxed{3}\ \boxed{3} = 6 \times 5 \times 4 \times 3 \times 3 = 1080$

$\times 3C$

(ii) **(3 points)** The digits 0, 1, 2, ..., 6 will be used to construct car plates, where each plate has 5 digits. If the first, third and forth digits must be even, the fifth digit is odd, and no repetition. How many car plates can be constructed ?

non repeat

$\boxed{4}\ \boxed{3}\ \boxed{3}\ \boxed{2}\ \boxed{3}$ odd

even

$= 4 \times 3 \times 3 \times 2 \times 3 = 216$

even left: 1
odd left: 2
Total even: 4
Total odd: 3
Total: 7

(iii) **(3 points)** The digits 0, 1, 2, ..., 6 will be used to construct car plates, where each plate has 5 digits. If the first, third and forth digits must be even, the fifth digit is odd, and repetition is allowed. How many car plates can be constructed?

repeat ✓

$\boxed{4}\ \boxed{7}\ \boxed{4}\ \boxed{4}\ \boxed{3}$ odd

even

$= 4 \times 7 \times 4 \times 4 \times 3 = 1344$

(iv) **(3 points)** Let m be the number of balls that will be distributed over 90 schools. What is the minimum value of m so that a school will have at least 81 balls.

$\left\lceil \frac{m}{90} \right\rceil = 81$

$\Rightarrow (80 \times 90) + 1 = \text{best} = 7201$

0 1 2 3 4 5 6 7

(v) **(3 points)** 579 positive ODD integers are available. Then there are at least m odd integers out of the given 579 ~~463~~ odd integers say $a_1, \ldots, a_m$ such that $a_1 \ (mod\ 8) = a_2\ (mod\ 8) \cdots = a_m\ (mod\ 8)$. What is the best value of m?

$\left\lceil \frac{579}{4} \right\rceil = 145 = m$

$c = \{1, 3, 5, 7\}$ $n = \{1, 3, \ldots, 579\}$

(vi) **(3 points)** Convince me that $2x^7 + 60x^4 + 15X^3 + 90 = 0$ has no rational numbers as a solution.

Leading coefficient

take 3

$3 \mid 90$ ✓, $3 \mid 15$ ✓, $3 \mid 60$ ✓

but $3 \nmid 2$

hence $2x^7 + 60x^4 + 15x^3 + 90 = 0$ cannot hold a rational root since 3 is not factor of 2.

2 ✗

QUESTION 4. (6 points) Consider the function $f(x) : [-8, -4] \to [0, 16]$ such that $f(x) = (x + 4)^2$.

(i) Sketch $f(x)$

$f(x): [-8,-4] \to [0,16]$

$(x+4)^2 = y$

$x=0 \Rightarrow y=16$

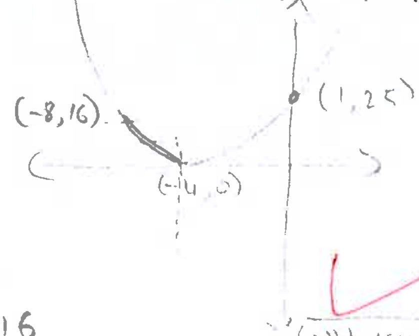
$x=1 \Rightarrow y=25$

$x=-1 \Rightarrow y=9$

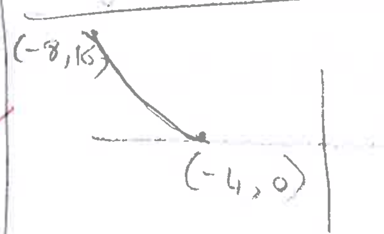
$x=-4 \Rightarrow y=0$

$x=-7 \Rightarrow y=9$

$x=-8 \Rightarrow y=16$



graph for $f(x)$



(ii) Is $f(x)$ ONTO? Is $f(x)$ one-to-one? you may use the staring-method and some tests.

onto: codomain $= [0,16]$

range $= [0,16]$

$\Rightarrow$ codomain = range $\Rightarrow$ it is onto ✓

one-to-one: from horizontal line test, we can see each element y in range has exactly one x in domain such that $y = f(x)$

(iii) If f^{-1} exists, find f^{-1}. Then find its domain, and its co-domain.

Since f is one-to-one & onto, f^{-1} exists ✓

$f^{-1} = ?$

$(x+4)^2 = y$

$x + 4 = \pm\sqrt{y}$

$x = \pm\sqrt{y} - 4$

we choose $-$ve sign since the f is defined there, domain.

$x = -\sqrt{y} - 4$

domain $= [0,16]$

co-domain $= [-8,-4]$

$x(0) = -4$

$x(16) = -8$

QUESTION 5. (6 points) Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 7 & 8 & 6 & 5 & 9 & 2 & 1 \end{pmatrix}$

a) Find the smallest integer n such that $f^n = I(x)$

we find the cycles:

$f = (1\ 3\ 7\ 9) \circ (2\ 4\ 8) \circ (5\ 6)$

4-cycle, 3-cycle, 2-cycle

$LCM(4,3,2) = 12$, $n = 12$

b) Find the smallest integer m such that $f^m = f^{-1}$, and find f^{-1}

$f^m = f^{-1}$

$f^n = I(x) \cdot f$

$f^m = n - 1 = 11$

$f^{-1} = ? = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 1 & 2 & 6 & 5 & 3 & 4 & 7 \end{pmatrix}$